MATH 286 HANDOUT 2: EXISTENCE AND UNIQUENESS THEOREM FOR A FIRST ORDER LDE

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A first order linear differential equation satisfies a stronger existence and uniqueness theorem than a non-linear differential equation:

Theorem 1. *Let $g(t)$, $r(t)$ be functions continuous in an interval (a, b) and let t_0, y_0 be numbers, $a < t_0 < b$. Then there is one and only one function $y = y(t)$ defined for $a < t < b$ which satisfies the initial value problem*

$$y' = g(t)y + r(t),\ y(t_0) = y_0. \tag{1}$$

This means that, if $g(t)$, $r(t)$ are continuous functions defined on a domain D which is a union of disjoint open intervals (if $g(t)$, $r(t)$ are defined on different domains, then D is the intersection of their domains), then the domain of the solution of the initial value problem (1) is the largest open interval contained in D which contains the point t_0! This is generally not true for non-linear differential equations.

Example: Find the domain of the solution y of the initial value problem

$$y' = \frac{e^t}{1+t}y + \ln|t|,\ y(-1/2) = 2000.$$

Remark: You do not have to solve this initial value problem in order to answer this question!
Solution: We have $g(t) = \frac{e^t}{1+t}$, $r(t) = \ln|t|$. The domain of $g(t)$ is $(-\infty, -1) \cup (-1, \infty)$. The domain of definition of $r(t)$ is $(-\infty, 0) \cup (0, \infty)$. So D is the intersection of these domains, which is $(-\infty, -1) \cup (-1, 0) \cup (0, \infty)$. We have $t_0 = -1/2$, so the largest open interval containing t_0 and contained in D is $(-1, 0)$. So the domain of y is $(-1, 0)$.

The existence and uniqueness theorem for a first order LDE is easy to prove, because it follows from the formula for solving a first order LDE. But in the above example we saw that we can use the theorem even without figuring the formula explicitly.