MATH 286 HANDOUT 4: SEPARATION OF VARIABLES FOR AUTONOMOUS SYSTEMS OF TWO ODE'S

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Suppose we have an autonomous system of two ODE's:

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y). \end{aligned} \tag{1}$$

We can write this as

$$\begin{aligned} \tfrac{dx}{dt} &= f(x,y) \\ \tfrac{dy}{dt} &= g(x,y). \end{aligned} \tag{2}$$

But in (2), it looks like we can divide one equation by the other. For example, we may try to define the second equation by the first one and get

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}. \tag{3}$$

(This can be made precise using the chain rule of differentiation.) But we can then treat this as a single first order ODE, with y regarded as a function of x, so we have converted the problem (1) to the simpler problem (3).

In order for this to work, we must have

$$\frac{dx}{dt} \neq 0,$$

which means

$$f(x,y) \neq 0.$$

Alternately, if $g(x,y) \neq 0$, we can proceed analogously, dividing the first equation (2) by the second equation. However, what if

$$f(x,y) = g(x,y) = 0?$$

Then at the point (x,y), we have an equilibrium (=constant) solution of the system (1). So, we understand those points as well.

Example 1:

$$\begin{aligned} y' &= xy^2 \\ x' &= x^2 y. \end{aligned} \tag{4}$$

Solution: Write the system in the form

$$\frac{dy}{dx} = \frac{y}{x},$$

which gives

$$y = Kx. \tag{5}$$

From the second equation (4), we then get

$$x' = Kx^3,$$

which gives

$$-\frac{1}{2x^2} = Kt + C$$

or

$$x = \pm\frac{1}{\sqrt{D - 2Kt}},$$

which then gives

$$y = \pm\frac{K}{\sqrt{D - 2Kt}}$$

(same sign). Note carefully that we are *not* allowed to absorb the constant K into other constants when solving for x, because the constant occurs in (5).

Example 2: The autonomous second order ODE:

$$y'' = f(y, y'). \tag{6}$$

We rewrite (6) as a first order system

$$\begin{aligned} y' &= x \\ x' &= f(y, x). \end{aligned} \tag{7}$$

So we have

$$\frac{dx}{dy} = \frac{f(y, x)}{x}$$

which we can solve to find x in terms of y, then plug into the first equation (7) to find y in terms of t.

Example 3: Using the general method of the previous example, solve:

$$y'' = \frac{(y')^2}{y}.$$

Solution: rewrite the equation as the system

$$\begin{aligned} y' &= x \\ x' &= \tfrac{x^2}{y}. \end{aligned} \tag{8}$$

This gives

$$\frac{dx}{dy} = \frac{x}{y},$$

or

$$x = Ky.$$

From the first equation (8), we then find

$$y' = Ky,$$

which gives

$$y = Ce^{Kt}.$$