MATH 286 HANDOUT 6: FROM INTEGRATING FACTORS TO VARIATION OF CONSTANTS

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We shall learn how to solve a non-homogeneous linear system of ODE's

$$Z' = AZ + R \tag{1}$$

where

$$Z = \begin{pmatrix} z_1 \\ \dots \\ z_n \end{pmatrix}, \; A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}, \; R = \begin{pmatrix} r_1 \\ \dots \\ r_n \end{pmatrix},$$

z_i are unknown functions of one variable t, and a_{ij} and r_i are known functions of t (not necessarily constant), provided we know the general solution to the corresponding homogeneous system

$$Y' = AY. \tag{2}$$

We have learned that to do this, it suffices to find any one particular solution Z of the system (1): the general solution is then of the form

$$Z + Y$$

where Y is a general solution of the system (2).

But how do we find even one particular solution Z? There is one case where we can do it: when $n = 1$. Then we just have a single linear differential equation

$$z' = az + r. \tag{3}$$

Remember, we solved this by an integrating factor μ: multiplying (3) by μ, we get

$$z'\mu - a\mu z = r\mu, \tag{4}$$

so if

$$\mu' = -a\mu, \tag{5}$$

then (4) becomes just

$$(\mu z)' = r\mu, \tag{6}$$

which we can solve. But what is familiar about the equation (5)? If we change the sign on the right hand side, we get

$$y' = ay, \tag{7}$$

which is the homogeneous equation corresponding to (3). But how do the solutions of the two equations (5) and (7) relate? You can easily check that the relationship is

$$\mu = 1/y.$$

So, how is the method of integrating factors expressed in terms of y? You see that (6) becomes just

$$(z/y)' = r/y,$$

where y is already known from (7). Another way to say this is as follows: we are seeking a solution z in the form

$$z = Ky, \tag{8}$$

where K is not a constant, but a function; then (6) becomes just

$$K' = r/y.$$

In this form, the method is called *variation of constants*, because the formula (8) looks the same as the formula for a general solution of the homogeneous equation (7), except that K is a *function* not a constant.

The variation of constants point of view has some advantages: for one thing, it uses the solution of the homogeneous equation, which we are often interested in finding anyway. The other point is that this method generalizes to systems almost mechanically: suppose we have a basis of the vector space of solutions of the system (2). Now form an $n \times n$ matrix Y whose columns are the basis solutions: then the matrix Y also satisfies the system (2), if we interpret multiplication as matrix multiplication. Now using variation of constants, we seek a $n \times 1$ matrix (=column vector) of functions K such that

$$Z = YK \tag{9}$$

is the solution of the non-homogeneous system (1). Let's write (1), using the formula (9), with unknown K:

$$Y(K') + Y'K = AYK + R. \tag{10}$$

But by (2), $Y'K = AYK$, so (10) becomes

$$YK' = R,$$

or

$$K' = Y^{-1}R. \tag{11}$$

Here Y^{-1} is the matrix *inverse* to Y, which means a matrix such that $Y^{-1}Y = Y(Y^{-1}) = I$. It turns out that any square matrix with non-zero determinant has an inverse. But $det(Y)$ is the Wronskian, so it is non-zero at every time t! The inverse of a matrix Y with non-zero determinant is given as follows: the element u_{ij} in the i'th row and j'th column of Y^{-1} is

$$u_{ij} = (-1)^{i+j} det(Y_{ji})/det(Y) \tag{12}$$

where Y_{ij} is the matrix obtained from Y by deleting the i'th row and j'th column. The formula (12) is called the *Cramer rule*, and can be easily checked using the row expansion formula for determinants. Notice the switching of the indices j, i on the right hand side of (12). For example, the inverse of a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with non-zero determinant $D = ad - bc$ is

$$\begin{pmatrix} d/D & -b/D \\ -c/D & a/D \end{pmatrix}.$$

Check that!

Example: Solve:

$$Z' = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} Z + \begin{pmatrix} e^t \\ 1 \end{pmatrix} \tag{13}$$

Solution: The corresponding homogeneous system

$$Y' = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} Y$$

has basis solutions

$$\begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix}, \begin{pmatrix} 2e^{3t} \\ 3e^{3t} \end{pmatrix},$$

so we can form the matrix

$$Y = \begin{pmatrix} e^{-2t} & 2e^{3t} \\ -e^{-2t} & 3e^{3t} \end{pmatrix}.$$

The inverse matrix is

$$Y^{-1} = \begin{pmatrix} 3e^{2t}/5 & -2e^{2t}/5 \\ e^{-3t}/5 & e^{-3t}/5 \end{pmatrix}.$$

So the equation for K is

$$K' = Y^{-1}R = \begin{pmatrix} 3e^{3t}/5 - 2e^{2t}/5 \\ e^{-2t}/5 + e^{-3t}/5 \end{pmatrix}.$$

So we get

$$K = \begin{pmatrix} e^{3t}/5 - e^{2t}/5 \\ -e^{-2t}/10 - e^{-3t}/15 \end{pmatrix},$$

and

$$Z = YK = \begin{pmatrix} -1/3 \\ -e^t/2 \end{pmatrix}.$$

This is a particular solution, the general solution is

$$Z = \begin{pmatrix} -1/3 \\ -e^t/2 \end{pmatrix} + C \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix} + D \begin{pmatrix} 2e^{3t} \\ 3e^{3t} \end{pmatrix}.$$

Remark: The simple form of the particular solution suggests that it could perhaps be guessed. This is true in many cases (including this example), and we will learn some such “educated guesses”. However, the method of variation of constants is, at least in principle, completely general.