MATH 286 HANDOUT 1: TWO BASIC SUBSTITUTIONS

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1. The linear substitution. Suppose we have a differential equation

$$y' = f(ax + by)$$

where f is a known function. Substitute

$$z = ax + by.$$

We have

$$z' = a + by' = a + bf(z).$$

So, we got $z' = a + bf(z)$, which is an autonomous (therefore separated) equation. Do not forget, after solving for z, to substitute back to get an answer for y.

Example: To solve

$$y' = (x + y)^2,$$

substitute $z = x + y$. So we get

$$z' = 1 + y' = 1 + z^2.$$

So we can solve for

$$z' = 1 + z^2.$$

We get

$$z = \tan(x + C),$$

so the final answer is

$$y = \tan(x + C) - x.$$

2. The homogeneous substitution. Suppose we have a differential equation

$$y' = f\left(\frac{y}{x}\right)$$

(called the *homogeneous equation*). In this case, substitute

$$z = \frac{y}{x},$$

so

$$z' = \frac{y'x - y}{x^2} = \frac{y' - z}{x} = (f(z) - z) \cdot \frac{1}{x}.$$

So we got

$$z' = (f(z) - z) \cdot \frac{1}{x}$$

which is a separable equation. Solve and then substitute $y = zx$ for a final answer.

Example:

$$y' = \frac{y^2 + xy + x^2}{yx + x^2}.$$

Rewrite as

$$y' = \frac{(\frac{y}{x})^2 + (\frac{y}{x}) + 1}{(\frac{y}{x}) + 1}.$$

So

$$f(z) = \frac{z^2 + z + 1}{z + 1},$$

$$f(z) - z = \frac{1}{z + 1},$$

so the differential equation becomes

$$z' = \frac{1}{1 + z} \cdot \frac{1}{x}.$$

Separating, we get

$$(z + 1)dz = dx/x,$$

so

$$z^2/2 + z = \ln|x| + C.$$

Solving the quadratic, we get

$$z = -1 \pm \sqrt{D + 2\ln|x|}.$$

Substituting back,

$$y = -x \pm x\sqrt{D + 2\ln|x|}.$$