Example 1 5h p. 37
A deck of 52 cards is dealt to 4 players.
(a) What is the probability one of the players receives all 13 aces?
(b) What is the probability each player receives one ace?

(a) Solution: E₁, E₂, E₃, E₄
Eᵢ means: Player i has all 13 aces.
\[ P(E_i) = \frac{1}{\binom{52}{13}} \]

\[ E_i \cap E_j = \emptyset \quad \text{if} \quad i \neq j \]

\[ P(E_1 \cup E_2 \cup E_3 \cup E_4) = \frac{4}{\binom{52}{13}} = 6.3 \cdot 10^{-12} \]

(b) **Solution**: How many ways are there to deal 52 cards among 4 players (each receive 13)?

\[
\binom{52}{13, 13, 13, 13} = \frac{52!}{13! \cdot 13! \cdot 13! \cdot 13!}
\]

If we deal the cards without the aces, the number is
\[
\binom{48}{12 \ 12 \ 12 \ 12}.
\]

Now deal the aces one to each player

\[
\binom{4}{1 \ 1 \ 1 \ 1} = 4!.
\]

The answer

\[
\frac{4! \cdot \frac{48!}{(12!)^4}}{\frac{52!}{(17!)^4}} = \frac{4! \cdot 13^4}{52 \cdot 51 \cdot 50 \cdot 49} \approx 0.1055
\]
Example 2: 5: p. 38

If n people are in a classroom, what is the probability that no two celebrate their birthday on the same day of the year?

Solution: \[ |E| = (365)^n \]

\[ |E| = 365 \cdot 364 \cdot \ldots \cdot (365 - m + 1) \]

\[ P(E) = \frac{365 \cdot 364 \cdot \ldots \cdot (365 - m + 1)}{(365)^m} \]

Turns out, this is \( < \frac{1}{2} \) when \( m \geq 23 \).
If \( n = 50 \), the probability that two people have their birthday on the same day of the year is \( \approx 0.97 \)

\[ n = 100 \quad \quad 2,000,000 : 1 \]

**Example 3:** 5j on p. 38

Shuffle 52 cards and turn the cards face up one at a time until the first ace appears. Is it more likely the next card will be the ace of spades or the two of clubs?
Solution: \( |S| = 52! \)

\[ |E_{AQ} | = 51! \]

Let the A Q out. Shuffle the remaining cards

51! possibilities

Now insert the A Q randomly.

52 choices

There is precisely one choice in \( E_{AQ} \): when A Q is inserted after the first ace.

\[ P(E_{AQ}) = \frac{1}{52} = P(E_{2 \cdot \phi_{52}}). \]
Example: p. 40 In a club,

36 members play tennis
28 play squash
18 play badminton
22 play tennis & squash
12 play tennis & badminton
9 play squash & badminton
4 play all three.

How many people play at least one of these sports?

Answer: $36 + 28 + 18 - 22 - 12 - 9 + 4 = 43$. 
Example 3: N men at a party each lose their hat.
At the end, each randomly select a hat. What are the chances nobody get their own hat?

Solution: \( E_i = \{ \text{the } i^{th} \text{ person did get their own hat back} \} \)

\[ |S| = N! \]
\[ |E_i| = (N-1)! \]
\[ p(E_i) = \frac{(N-1)!}{N!} \]

\[ i < \ldots < i_k \]
\[ |E_{i_1} \cap \ldots \cap E_{i_k}| = (N-k)! \]
\[
\begin{align*}
\binom{N}{k} P(E_1 \cap \cdots \cap E_k) &= \binom{N-k}{k} \\
\quad \text{for } k = 1, 2, \ldots, N-1 \\
\end{align*}
\]

\[
P(E_1 \cup \cdots \cup E_N) = N \cdot \frac{(N-1)!}{N!} - \frac{(N)^2}{N!} \left( \frac{(N-2)!}{N!} \right) + \cdots + \left( \frac{(-1)^{k-1}(N)^{N-k}!}{N!} \right) + \cdots + (-1)^{N-1} \left( \frac{N^{0}!}{N!} \right)
\]

\[
\binom{N}{k} \left( \frac{(N-k)!}{N!} \right) = \frac{N!}{k!(N-k)!} \left( \frac{(N-k)!}{N!} \right) = \frac{1}{k!}
\]
\[ p(E_1 u \ldots u E_N) = 1 - \frac{1}{2!} + \frac{1}{3!} - \ldots + (-1)^{N-1} \frac{1}{N!} \]

Answer: \[ 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^{N-1} \frac{1}{N!} \]

\[ \lim_{N \to \infty} \left( \sum_{k=0}^{\infty} \frac{x^k}{k!} \right) = e^x \approx 0.37. \]

\[ \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x \]

\[ e^x \cdot e^y = \left( \sum_{k=0}^{\infty} \frac{x^k}{k!} \right) \left( \sum_{l=0}^{\infty} \frac{y^l}{l!} \right) = \]
\[
= \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{1}{k!} \cdot \frac{1}{(m-k)!} x^k y^{m-k} = \\
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) x^k y^{n-k} = \sum_{n=0}^{\infty} \frac{1}{n!} (x+y)^n
\]

\[
(x+y)^m = e^{x+y}
\]