CONWAY NUMBER GAMES FOR MULTIPLE PLAYERS

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1. Introduction

There are many different mathematical meanings of the word 'game'. Regardless of the kind of games we consider, people agree that games of n players are much more difficult to understand for $n > 2$ than for $n \leq 2$. In this paper, we consider deterministic games, i.e. games where each player in each position has a well defined set of moves, which, in a fixed way, change the position to another position (in fact, it is clear that there is no point in distinguishing between positions and games, so we can substitute the word 'game' for the word 'position' everywhere). This, of course, describes only the 'static' aspect of the rules of a game. The 'dynamic' aspects refer to how the game is actually played. A play by play sequence of moves in a game will be called a 'match'. The dynamical rules of matches which we will consider specify a certain order of the set of players; the players shall move repeatedly in the same order of play until a certain player cannot move, at which point the match shall end. The player who cannot move shall then be declared the loser of the match. We shall consider only games where there is no possibility of infinite matches.

Even for such deterministic games, however, it is difficult to make any conclusions about the course of matches for $n > 2$. The reason is that unlike the case of $n = 2$, there is no natural order of preference of the outcomes of the game from the point of view of the i'the player. While the i'th player obviously prefers not to lose, there is no natural reason why he should a priori prefer one particular other player to lose. Yet, such preferences will determine strategies, and ultimately the outcome of a match. Preferences can even change throughout the course of a match. Thus, it is usually said that few strategic conclusions about deterministic games of $n > 2$ players can be made without introducing non-deterministic concepts, perhaps even non-mathematical concepts (e.g. psychology).

The purpose of this paper is to look at a certain very special class of deterministic games of n players, for which certain strategical conclusions can be made in a rigorous mathematical setting, without introducing outside concepts. The motivation for introducing our particular class of games is that they generalize the 'number games' for 2 players from J.H.Conway's famous book [1]. For this reason, we shall call this class of games *Conway number games for n players.*

Conway number games for 2 players are games to which one can assign a value which is a 'number'. Here the word number means element of a certain ordered field, known as the Conway field $\mathbb{C}$. The Conway field contains, among other things, all ordinal numbers, as well as any other ordered field: it is a foundational-level object of set theory (in fact, Conway introduces his own approach to formal set theory based on number games in [1]).

However, this is not the aspect of number games which we will be most interested in. Rather, the main point of number games is that ‘no player can possibly improve his own position by making a move’. This, of course, needs precise definition, (in particular in reference to the word ‘improve’). Definitions will be provided later. We shall, however, remark here that one could adopt the point of view that for this reason, number games are generally strategically unintersting, since the very meaning of strategy is being able to take advantage of one’s own move in the best possible way. We would like to remark here that in our opinion, that conclusion is not correct; number games are interesting from a certain, very special, strategic point of view. Namely, they represent *purely consumer-based models*: in other words, we can view these games as models where each player consumes his own fixed assets, with the object of ‘outlasting’ the other player, i.e. avoiding “dying first”.

With this in mind, it becomes interesting to try to define analogues of Conway number games for n players, and analyse what, if any, strategic conclusions one can make for such games. In this paper, we do give one possible definition of such number games for n players. The set of (equivalence classes) such games is an $n-1$-dimensional vector space over the Conway field (to avoid set-theoretical issues, we restrict attention to real numbers).

On the positive side, the set of number games of n players has many of the formal properties of number games of 2 players, it is a non-trivial fact that such a theory can “work” at all. Also, we shall be able to make certain strategic conclusions for number games: in particular, each match will have a well defined ‘loser’ who can always be defeated if all the other players act ‘in concert’. Note, however, that this statement is weaker than the corresponding statement for $n = 2$, and does not provide a *definitive* strategic analysis of the game. Also on the negative side, our definition of number games of n players is somewhat unnatural (or more precisely, its naturality becomes clear only in the proofs of the main properties of such games). Also, numbers, as well as Conway games of 2 players are involved in our definition of number games of n players recursively. Because of this, our Conway games for n players are less ‘fundamental’ than in the case of $n = 2$. However, it is possible that this could be remedied by finding a better definition.

This paper is organized as follows: In the next section, we shall present basic definitions and facts about games and matches which do not involve numbers. In Section 3, we shall introduce number games, and prove our main results.

2. GAMES AND MATCHES

In this paper, a *game with set T of players* is defined recursively as follows:

1. The empty set $\emptyset$ is a game (also called 0).
2. If G_i are games for all $i \in T$, then the tuple $(G_i)_{i\in T}$ is a game.
3. Every game can be obtained by 1, 2 in a possibly transfinite number of steps.

Obviously, only the cardinality of the set T matters. We shall mostly consider the case $T = \{1, ..., n\}$ (in which case we shall simply speak of *games of* n *players*), but it is useful to allow other T's, notably $T \subset \{1, ..., n\}$.

We shall now introduce our main strategic concept for games of n players. It is important to notice that this concept does not involve dynamic aspects of games, i.e. matches.

Specifically, we shall inductively define

$$G <_S 0$$

for a set $S \subseteq T$ if the following conditions hold:

1. If $i \in S$ then for all $H \in G_i$, $H <_{\{i\}} 0$.
2. If $i \notin S$ and $j \in S$ then there exists an $H \in G_i$ and a set U with $j \in U \subseteq S \cup \{i\}$, such that $H <_T 0$.

We shall write

$$G \sim 0$$

if $G <_T 0$.

Lemma 1. *For each game* G*, there exists at most one* S *such that* $G <_S 0$.

Proof: Induction: Note that

$$0 <_S 0 \text{ if and only if } S = T.$$

Assume the statement true for all $H \in G_i$ for all $i \in T$. Then if

$$G <_S 0,$$

note that $i \notin S$ if and only if there exists a $H \in G_i$ such that it is not true that $H <_{\{i\}} 0$, which uniquely characterizes S. □

We now explain the dynamical significance of $G <_S 0$. To this end, we must define matches. Assume now that T is finite, and that we have a bijection

$$\sigma : \{1, ..., n\} \to T.$$

Such bijection will be called a *dynamic rule.* A match according to the dynamic rule σ is a sequence of games

$$(G(j))_{j=1,\dots,N}$$

where $G(1) = G$,

$$G(j+1) \in G(j)_{k(j)} \text{ where } k(j) = \sigma(j'),\ j' \equiv j \mod n,$$

$$G(N)_{k(N)} = \emptyset.$$

Then $k(N)$ is called *the loser of the match.* Note that

$$(G(j))_{j=2,\dots,N}$$

is a match according to the dynamic σ'-rule where

$$\sigma'(j) = \sigma(j+1) \text{ for } j < n,$$

$$\sigma'(n) = \sigma(1).$$

Note also that by part 3 of our definition of game, it is impossible to have an infinite match, i.e. an infinite sequence satisfying the properties of a match without the N. (Proof: induction.)

We now define inductively our main dynamic strategic concept. A player i is called the *loser of a game G according to the dynamic rule σ* if

1. If $\sigma(1) = i$ then for all $H \in G_i$, i is the loser of H according the the dynamic σ'-rule.
2. If $\sigma(1) \neq i$, then there exists an $H \in G_{\sigma(1)}$ such that i is the loser of H according to the dynamic σ'-rule.

Intuitively speaking, this means that i will lose any match according to the dynamic rule σ, provided that all the other players act "in concert".

Proposition 2. *Suppose $G <_S 0$ and suppose that σ is any dynamic rule. Let j be minimal such that $\sigma(j) \in S$. Then $\sigma(j)$ is the loser of the game G according to the dynamic rule σ.*

Proof: Induction. If $\sigma(1) = i$, then for all $H \in G_i$, $H <_{\{i\}} 0$, so the induction hypothesis applies. If $\sigma(1) \neq i$, then there exists an $H \in G_{\sigma(1)}$ such that $H <_U 0$ for some $i \in U \subseteq S \cup \{\sigma(1)\}$. Note that i satisfies the induction hypothesis with G replaced by H, and σ replaced by σ'. $\square$

For the purposes of the next section, we shall now define the *sum of games*: Define inductively

$$G + H$$

by

$$(G + H)_i = \{G + K | K \in H_i\} \cup \{K + H | K \in G_i\}.$$

3. Number games

We continue to assume that T is finite, of cardinality n. We shall work with T-tuples of real numbers (or more generally T-tuples of elements of any ordered field F)

$$g = (g_i)_{i \in T} \tag{1}$$

which satisfy

$$\sum_{i \in T} g_i = 0.$$

Obviously, the set of all such T-tuples is an $n-1$-dimensional vector space over F, which we shall denote by F_T. For $S \subseteq T$, and for the T-tuple (1), we now write

$$g <_S 0 \tag{2}$$

if

$$g_i < g_j \text{ for all } i \in S, j \notin S.$$

Note that obviously for a given g, (2) happens for precisely one set S. We also write

$$g \leq_S 0$$

if $g <_U 0$ for some $U \supseteq S$. Note that $g <_T 0$ is equivalent to $g \leq_T 0$ which is equivalent to $g = 0$. We shall write

$$g <_S h$$

if $g - h <_S 0$, and similarly for $\leq_S$. By abuse of notation, we write $<_i$ instead of $< \{i\}$.

Lemma 3. *If $g <_i h$, then $g_i - g_j < h_i - h_j$ for all $j \neq i$.*

Proof: $g <_i h$ means $g - h <_i 0$, i.e. $g_i - h_i < g_j - h_j$ for all $j \neq i$. □

Below, we shall need the following construction. For $i \in T$, consider the function

$$p_i : F_T \to F_{T-\{i\}}$$

given by

$$p_i(g) = \left(g_j + \frac{g_i}{n-1}\right)_{j \in T-\{i\}}.$$

We now proceed to number games. We begin by recalling briefly Conway number games of 2 players [1]. The main point is that to each pair of subsets

$$\langle A|B\rangle$$

of the Conway field $\mathbb{C}$, such that for all $x \in A$, $y \in B$ we have

$$x < y,$$

there is assigned an element

$$v\langle A|B\rangle \in \mathbb{C}$$

such that, for all $x \in A$, $y \in B$,

$$x < v\langle A|B\rangle < y.$$

We refer the reader to [1] for details, but the following two properties are crucial for our purposes:

1. $v(\emptyset) = 0$ and $v(G + H) = v(G) + v(H)$ where
$$\langle A|B\rangle + \langle C|D\rangle = \langle v\langle A|B\rangle + C|v\langle C|D\rangle + B\rangle.$$
2. If $C \supseteq A$ and $D \supseteq B$ and for each $x \in C$ (resp. $y \in D$) $x < v\langle A|B\rangle$ (resp. $v\langle A|B\rangle < y$) then
$$v\langle C|D\rangle = v\langle A|B\rangle.$$

Using this, we define inductively a *number game of T players* as a game G for which there exists an n-tuple

$$v(G) \in \mathbb{C}_T$$

such that

1. For all $i \in T$ and for all $H \in G_i$, H is a number game and $v(H) <_i v(G)$.
2. For all $i \neq j \in T$,
$$\begin{aligned} v_i(G) - v_j(G) = \ & v\langle \{v_i(H) - v_j(H) | H \in G_i,\ p_i v(H) \leq_j p_i v(G)\} | \\ & \{v_i(H) - v_j(H) | H \in G_j,\ p_j v(H) \leq_i p_j v(G)\}\rangle. \end{aligned}$$

Lemma 4. *The T-tuple $v(G)$, if it exists, is uniquely determined.*

Proof: By (1) of the definition of number game and Lemma 1, for all $H \in G_i$ we have

$$v_i(H) - v_j(H) < v_i(G) - v_j(G)$$

while for all $H \in G_j$ we have

$$v_i(H) - v_j(H) > v_i(G) - v_j(G).$$

By property (2) of Conway games, (2) of the definition of number games then implies

$$v_i(G) - v_j(G) = v\langle\{v_i(H) - v_j(H) | H \in G_i\}|\{v_i(H) - v_j(H) | H \in G_j\}\rangle \tag{3}$$

which recursively determines $v(G)$. □

Corollary 5. *(of (3))*

$$v(G + H) = v(G) + v(H).$$

□

Lemma 6. *A sum of number games is a number game.*

Proof: By induction, both conditions (1), (2) are obviously additive. In particular, in (2), the right hand side for a sum of games contains the Conway sum of the righr hand sides of (2) of the individual games, so we can use properties (1) and (2) of Conway games. □

Proposition 7. *If G is a number game and $v(G) <_S 0$, then $G <_S 0$.*

Proof: Induction. By the induction hypothesis, condition (1) in the definition of number games implies condition (1) for $G <_S 0$.

Suppose condition (2) for number games is valid for G. Choose $i \notin S$, $j \in S$. Then, by definition of $v(G) <_S 0$,

$$v_i(G) > v_j(G).$$

By (2) for number games and properties of Conway games, there is an $H \in G_i$ such that

$$v_i(H) \geq v_j(H),\ p_i v(H) \leq_j p_i v(G). \tag{4}$$

The second condition implies that

$$v_j(H) - v_k(H) \leq v_j(G) - v_k(G) \leq 0 \text{ for all } k \neq i, j, \tag{5}$$

so together with (4) this implies that

$$v_j(H) = min\{v_p(H) | p \in T\}.$$

On the other hand, if $k \notin S$, (5) implies that

$$v_j(H) < v_k(H).$$

Thus, $H <_T 0$ for some $j \in T \subseteq S \cup \{i\}$, as required in condition 2 for $G <_S 0$. □

Remark: Since for $g \in \mathbb{C}_T$, there is always a unique $S \subseteq T$, $S \neq \emptyset$ with $g <_S 0$, the converse of the Proposition is also true.

Corollary 8. *If G is a number game and $v(G)=0$ then $G \sim 0$.*

□

Corollary 9. *A number game G has an inverse, i.e. a game H such that $G+H \sim 0$.*

Proof: The symmetric group Σ_T obviously acts on number games by permuting players. Now we obviously have

$$v\left(\sum_{\sigma\in\Sigma_T} \sigma G\right) = \sum_{\sigma\in\Sigma_T} \sigma v(G) = 0,$$

so

$$\sum_{\sigma\in\Sigma_T} \sigma G \sim 0$$

by the previous Corollary. □

Proposition 10. *(Existence theorem) For every $g \in \mathbb{R}_T$ there exists a number game G with*

$$v(G) = g.$$

Comment: This is true with $\mathbb{R}$ replaced by the Conway field $\mathbb{C}$; we restrict to the real case here to avoid set-theoretical difficulties.

Proof: First note that the game G^i where

$$G^i_i = \{0\},$$
$$G^i_j = \emptyset \text{ for } j \neq i$$

is a number game with

$$v(G^i) = v^i$$

where

$$v^i_i = (n-1)/n,\ v^i_j = -1/n \text{ for } j \neq i. \tag{6}$$

The set of all $v(G)$ where G is a sum of copies of the G^i's therefore forms a lattice L spanned by (6). We shall call such games G lattice games.

Now let H be a lattice game. Select $i \in T$. Define G so that

$$G_i = \{\text{all lattice games} \leq_i H\}$$
$$G_j = \{\text{all lattice games} \leq_j H + G^j\} \text{ for } j \neq i.$$

Then it is easily verified that G is a number game and

$$v(G) = v(H) + \frac{1}{2}v^i.$$

In this fashion, we can obtain number games G where $v(G)$ is any element of $\frac{1}{2}L$. Repeating this procedure, we can get number gamer G where $v(G)$ is any element of $\frac{1}{2^N}L$ for any natural number N. Call such games dyadic games.

Now let $g \in \mathbb{R}_T$. Then define G by

$$G_i = \{\text{all dyadic games } H \text{ such that } v(H) <_i g\}.$$

Then G is a number game with value g. $\square$

REFERENCES

[1] J.H.Conway: *On numbers and games. Second edition*, A K Peters, Ltd., Natick, MA, 2001. xii+242 pp. ISBN: 1-56881-127-6

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