**SOME REMARKS ON MOTIVIC HOMOTOPY THEORY OVER ALGEBRAICALLY CLOSED FIELDS**

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1. Introduction

The purpose of this note is to point out a surprising aspect of the 2-complete motivic homotopy theory over algebraically closed fields of characteristic 0. We shall follow the notational convention from [2] (analogous to Real-oriented homotopy theory), which means that for a motivic (generalized) cohomology theory, we write

\[ E^{k+\ell,\alpha}(X) = E^{k+\ell}X, \]

and similarly for homology.

One can then define \( BPGL \) and a motivic analog of the Adams-Novikov spectral sequence converging to the 2-completed motivic stable homotopy group \( \pi^{\text{Mot}}_\ast \). The \( E_2 \)-term is equal to the topological \( E_2 \)-term, but with dimensions shifted by twists, and tensored with \( \mathbb{Z}[\theta] \). The differentials mimic the topological differentials, but there is a difference in twist. An example is given by the following peculiar result:

**Theorem 1.** Over an algebraically closed field \( k \) of characteristic 0, the element \( \alpha_1^2 \eta_{3/2} \in \text{Ext}_{BPGL}^{5,15(1+\alpha)} \) represents a non-zero element

\[ x \in \pi^{\text{Mot}}_{10+15\alpha} S^0 \]

which maps to 0 in etale homotopy theory, and satisfies

\[ \theta x = 0. \]

Similarly inside the \( \alpha \)-family, if we denote by \( a_{4k} \in \pi^{\text{Mot}}_{4k+4k\alpha} \) resp. \( a_{4k+1} \in \pi^{\text{Mot}}_{4k+1+4k\alpha} \) the elements represented by \( \alpha_{4k} \), then

\[ a_{4k} \eta^m \neq 0, \quad a_{4k+1} \eta^m \neq 0 \]

for any positive integer \( k, m \), while

\[ a_{4k} \eta^2 \theta = a_{4k+1} \eta^2 \theta = \eta^4 \theta = 0. \]

**Remark:** Note that computing Morel’s Milnor-Witt ring [7], one easily sees that \( \eta^m \neq 0 \).

2-adic etale stable homotopy theory over an algebraically closed field turns out to be just topological stable homotopy theory \( \otimes \mathbb{Z}[\theta, \theta^{-1}] \). Because of this, Theorem 1 gives an example where motivic stable homotopy groups of spheres substantially deviate from the topological case, by producing an element which “forgets” to 0. Even though throughout this paper we work over a field of characteristic 0, it should nevertheless be remarked that Theorem 1 remains valid over a field of finite characteristic which is a sufficiently large prime, by the remarks of [8].
In order to prove the Theorem, we need to know the structure of algebraic cobordism over an algebraically closed field. This in turn needs the Adams spectral sequence, which needs the algebra of bistable operations in motivic cohomology (the motivic Steenrod algebra). This calculation was done by Voevodsky in the late 90’s, but not published at the time of the writing of this note. 1 To make the present note self-contained, we give the computation here in the case of an algebraically closed field (although the method works in greater generality). Namely, by investigating the structure of symmetric products, we prove

**Theorem 2.** (Voevodsky) Let \( k \) be an algebraically closed field. Denoting by \( H^\text{Mot} \) the \( H\mathbb{Z}/2 \)-motivic (co)homology spectrum over \( k \), and by \( \theta \) Tate twist (of dimension \( 1 - \alpha \)), the algebra of bistable operations \( H^\text{Mot} \circ H^\text{Mot} \) is generated as a \( \mathbb{Z}/2[\theta] \)-module by reduced power operations \( P^s \) (of dimension \( s(1 + \alpha) \)) and the Bockstein (of dimension 1).

By [9], we therefore have

**Corollary 3.** The dual motivic Steenrod algebra over an algebraically closed field \( k \) is given by

\[
H^\text{Mot}_* \circ H^\text{Mot} = \mathbb{Z}/2[\theta, \tau_0, \tau_1, \tau_2, \xi_1, \xi_2, ...]/ (\tau_i^2 = \xi_{i+1} \theta)
\]

where the dimensions of \( \xi_i, \tau_i \) are \((2^i - 1)(1 + \alpha)\), \((2^i - 1)(1 + \alpha) + 1\), respectively.

Using this, we can calculate the 2-completed algebraic cobordism groups:

**Theorem 4.** Over an algebraically closed field \( k \), the 2-completed algebraic cobordism groups are given by

\[
(MGL^2)_* = \mathbb{Z}_2[\theta, v_1, v_2, ...]
\]

where \( v_n \) has dimension \((2^n - 1)(1 + \alpha)\).

The present note is organized as follows: In Section 2, we discuss the main technical tool of our approach to Voevodsky’s result, the motivic transfer. In Section 3, we apply this technique to analyzing the motivic cohomology of the symmetric smash-powers of spheres. In Section 4, we prove our main theorems and discuss the motivic Adams-Novikov spectral sequence.

## 2. The Motivic Elmendorf Construction

In this paper, we discuss symmetric products of varieties, which are generally not smooth, so we work with the cd-h topology in finite schemes over a field \( k \).

**Definition:** A 1-point compactification \( X^* \) of a quasiprojective variety \( X \) embedded into a projective variety \( \overline{X} \) is \( \overline{X}/(\overline{X} - X) \).

**Examples:** \( S^{n(1+\alpha)} \) is a 1-point compactification of \( \mathbb{A}^n \). \( \bigwedge_d S^{n(1+\alpha)}/\Sigma_d \) is a 1-point compactification of \( \mathbb{A}^{nd}/\Sigma_d \).

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1After this note was written, [10] appeared on the K-theory archive; the present note was written independently of [10].
We should note that if $X$ is smooth and $\overline{X} - X = D_1 \cup \ldots \cup D_m$ where $D_i$ are divisors with normal crossings, then denoting by $Q$ the poset of non-empty subsets of $\{1, \ldots, m\}$ with respect to inclusion, the natural map

$$\operatorname{holim}_{s \in Q} \bigcap_{i \in S} D_i \to \overline{X} - X$$

is an equivalence, which gives a model of the 1-point compactification in the smooth category.

**Lemma 5.** Let $X^*$ be a 1-point compactification of a smooth quasiprojective variety $X$ of dimension $n$. Then, in the $\mathbb{A}^1$-stable category,

$$X^* \simeq \Sigma^{n(1+\alpha)} D(X^\xi)$$

where $D$ denotes Spanier-Whitehead dual and $X^\xi$ denotes the Thom spectrum of $X$ with respect to its virtual normal bundle of dimension 0.

Now the main point of this section is to consider “smooth models” of quotients of smooth varieties by finite groups $G$. In topology, there is the following method, known as the Elmendorf construction: The orbit category $\mathcal{O}$ is the category of $G$-orbits and $G$-equivariant maps. Equivalently, it is the category of subgroups and outer subconjugacies (i.e. subconjugacies modulo inner conjugacies in the source group). Then the fixed points $X^?$ of a space $X$ under subgroups of $G$ form a contravariant functor $\mathcal{O} \to \text{Spaces}$. An example of a covariant functor on the orbit category is $G/\?$. Another example of a covariant functor is $\mathcal{E}_F$ where $\mathcal{E}_F^H$ is a $G$-CW complex characterized up to $G$-homotopy equivalence by the property that

$$\mathcal{E}_F^H^K \simeq \ast \text{ for } K \text{ subconjugate to } H$$

$$\emptyset \text{ else.}$$

One can make $\mathcal{E}_F^H$ into a covariant functor on the orbit category, for example by taking the 2-sided bar construction

$$\mathcal{E}_F^H = B(\ast, \mathcal{O}, \mathcal{O}/?)$$

(Here $\mathcal{O}/H$ denotes the contravariant functor which assigns to $K$ the set of morphisms $K \to H$ in $\mathcal{O}$.) The notation $\mathcal{E}_F^H$ is explained by noting that the set $\mathcal{F}^H_H$ of subgroups subconjugate to $H$ is an example of a family, i.e. set of subgroups of $G$ closed under subconjugacy. Analogously to (1), one can define the classifying space of any family. Then, in topology, the natural map

$$\mathcal{E}_F^H \times_{\mathcal{O}} X^? \to X^G$$

is an equivalence for any $G$-CW complex $X$.

In $\mathbb{A}^1$-homotopy, (3) can still be true for a scheme (or $G$-space) $X$, but (1) does not characterize $\mathcal{E}_F^H$ up to $G$-$\mathbb{A}^1$-equivalence, and the construction (2) is usually wrong for the purposes of (3). An alternative construction which will work in the
case we are interested in (the symmetric products) is obtained as follows: Let $G$ act effectively on $d = \{1, \ldots, d\}$. Then set

$$E_{\mathcal{F}_H} = A^d_{\infty} - \bigcup_{K \notin \mathcal{F}_H} (A^d_{\infty})^K.$$  

For any construction of $E_{\mathcal{F}_H}$, one has (by induction on the strata)

**Lemma 6.** Let $X^{>H} = \bigcup_{K \notin \mathcal{F}_H} X^K$. If the collapse map

$$E_{\mathcal{F}_H} \wedge (X^H/X^{>H}) \to X^H/X^{>H}$$

is an $H$-$A^1$-equivalence, then (3) is an $A^1$-equivalence. In fact, more generally, for any subgroup $\Gamma \subseteq G$, the natural map

$$\Gamma \backslash G/\pi \times E_{\mathcal{F}_?} \times \mathcal{O} X? \to X/\Gamma$$

is a $\Gamma$-$A^1$-equivalence.

□

**Corollary 7.** Under the assumptions of Lemma 6, let $(H)$ be the full subcategory of $\mathcal{O}$ on groups conjugate to $H$ and let $\mathcal{B}(X)$ denote Bloch’s Chow chain complex on $X$ (resp. its natural extension to $A^1$-$G$-spaces). Then the transfer maps associated with free group action

$$t: \mathcal{B}(E_{\mathcal{F}_?} \times (H) X?) \to \mathcal{B}(\Gamma \backslash G/\pi \times E_{\mathcal{F}_?} \times (H) X?)$$

multiplied by the multiplicity factors $|\pi_H(\Gamma \cap N(H))|$ where $\pi_H: N(H) \to W(H) = N(H)/H$ is the projection fit via (6) together to define a transfer map in the stable category $t: \mathcal{B}(X/G) \to \mathcal{B}(X/\Gamma)$.

Further, there exists a filtration on $\mathcal{B}(X/\Gamma)$ such that the associated graded map of the composition

$$\mathcal{B}(X/\Gamma) \to \mathcal{B}(X/G) \to \mathcal{B}(X/\Gamma)$$

is multiplication by $|G/H|$.

In the last sentence of the Corollary, the filtration could probably be eliminated by more careful consideration, but it does not matter for our purposes.

3. **Symmetric products**

In this section, we are interested in the following example:

$$X = \bigwedge_d S^{n(1+\alpha)}$$

where $\Sigma_d$ acts by permutation of coordinates. We claim that condition (5) is satisfied when we set (4) with $\Sigma_d$ acting on $d$ by the standard permutation representation.
First of all, we note that the strata $X^H/X^{>H}$ are only non-trivial when
\[(8)\]
\[H = \Sigma_{d_1} \times \ldots \times \Sigma_{d_k},\]
d\[d_1 + \ldots + d_k = d\] where $\Sigma_d$ acts on $d_i$ by the standard permutation representation. Then $X^H/X^{>H}$ is the 1-point compactification of the pure stratum
\[(9)\]
\[X^H - X^{>H} = \mathbb{A}^n_k - \Delta\]
where $\Delta$ is the big diagonal (the union of elements with 2 or more coordinates coinciding). Using (4) and (9), we can then define an $\mathbb{A}^1$-homotopy inverse
\[(10)\]
\[\iota : \mathbb{A}^n_k - \Delta \to E\mathcal{F}_H \times (\mathbb{A}^n_k - \Delta)\]
of the natural projection
\[(11)\]
\[p : E\mathcal{F}_H \times (\mathbb{A}^n_k - \Delta) \to \mathbb{A}^n_k - \Delta\]
($p$ collapses the first coordinate to a point). To define $\iota$, use $Id$ for the second coordinate in the target of (10). For the first coordinate, we need a map
\[\mathbb{A}^n_k - \Delta \to \mathbb{A}^\infty - \bigcup_{K \notin \mathcal{F}_H} (\mathbb{A}^\infty)^K.\]
But this is obvious: simply send
\[
((x_{11}, \ldots, x_{n1}), \ldots, (x_{1k}, \ldots, x_{nk})) \\
\mapsto ((x_{11}, \ldots, x_{n1}, 0, 0, \ldots)^{d_1}, \ldots, (x_{1k}, \ldots, x_{nk}, 0, 0, \ldots)^{d_k}) \in (\mathbb{A}^\infty)^d.
\]
By definition, (11) is strictly left inverse to (10). To construct an $\mathbb{A}^1$-homotopy
\[(12)\]
\[ip \simeq Id,\]
first recall that we have the “Milnor trick” homotopy
\[k_t : \mathbb{A}^\infty \to \mathbb{A}^\infty,\]
\[k_t(x_1, x_2, \ldots) = (1-t)(x_1, x_2, \ldots) + t(0, \ldots, 0, x_1, 0, \ldots, 0, x_2, \ldots)\]
($n$ 0’s inserted before each coordinate). Clearly, $(k_t)^d$ restricts to an $H$-homotopy
\[\ell_t : E\mathcal{F}_H \to E\mathcal{F}_H\]
where $\ell_0 = Id$, $\ell_1 \subseteq (0 \times 0 \times \ldots \times \mathbb{A} \times 0 \times \ldots \times 0 \times \mathbb{A} \times \ldots)^d$. (12) then follows if we can construct a homotopy
\[ip \simeq \ell_1,\]
But for this purpose, we may now simply use
\[t(ip) + (1-t)\ell_1,\]
which completes the proof of our claim.

Now we see that an $\mathbb{A}^1$-homotopy inverse and for the map in the assumption of Lemma 6 is in our case obtained simply by 1-point compactifying (10), (12) in the $\mathbb{A}^n_k - \Delta$ coordinate. We obtain

**Proposition 8.** Let $S \subset \Sigma_d$ be the 2-Sylow subgroup obtained by the standard permutation representation of a product of wreath products of copies of $\mathbb{Z}/2$. Then there exists for $X$ in (7) a transfer map
\[\tau : \delta(X/\Sigma_d) \to \delta(X') = \bigwedge_{d} S^{n(1+\alpha)}/S\]
such that if we denote by $p : H(X') \to H(X/\Sigma_d)$ the natural projection, then $\tau p$ is a 2-complete equivalence.

**Proof:** By Corollary 7, there is a filtration such that on the associated graded pieces, $\tau p$ is the multiplication by the odd number $|\Sigma_d/S|$.

Because of Proposition 8, it now makes sense to examine the motivic (co)homology of $X'$ in what we call the stable range, i.e. the range

$$i + j \alpha$$

where $(2n \leq i + j << 4n)$.

**Proposition 9.** There exists a stratification of $X'$ where the quotients of strata are 1-point compactifications of the corresponding pure strata, and each pure stratum is of one of the following forms:

13) $(\mathbb{G}_m)^{\times \ell} \times \mathbb{A}^{n+k}, \ k << 2n,$

or

14) An $\mathbb{A}^m$-bundle $V$ on a smooth variety $Y$ where $m >> n$.

Furthermore, strata of the form (13) do not occur unless $d = 2^k$ for some integer $k$.

**Proof:** Induction. It suffices to consider $d = 2^k$, since otherwise $X'$ is the smash-product of the cases for $2^{k_i}$ where $k_i$ are places of the 1’s in the binary expansion of $k$.

Now the case $k = 1$ is obvious, so consider

15) $X' = (X'' \wedge X'')/(\mathbb{Z}/2)$

where $X''$ is stratified as in the statement of the Proposition. So first, we take the stratification of $X'' \wedge X''$ by the smash-product of two copies of the stratification of $X''$, and then take a $\mathbb{Z}/2$-quotient. If the pure strata of $X''$ are $A_1, \ldots, A_N$, then this gives pure strata of (15) of the form

16) $A_i \times A_j, \ i < j$,

17) $(A_i \times A_i)/(\mathbb{Z}/2)$.

The strata (16) are clearly of the form (14), the strata (17) may need to be further stratified.

More concretely, if $A_i$ is of the form (14), we may stratify (17) by taking the bundle $Sym^2(V)$ on $Y$, and then the $\mathbb{Z}/2$-quotient of the bundle induced from $V \times V$ on $(Y \times Y) - Y$. Both are clearly of the form (14).

Suppose, then, that $A_i$ is of the form (13). Once again, then, we have the pure stratum

18) $$((\mathbb{G}_m)^{\times \ell} \times (\mathbb{G}_m)^{\times \ell}) - (\mathbb{G}_m)^{\times \ell} \times_{\mathbb{Z}/2} (\mathbb{A}^{n+k} \times \mathbb{A}^{n+k})$$

which is of the form (14). What we are left with is

19) $$(\mathbb{G}_m)^{\times \ell} \times Sym^2(\mathbb{A}^{n+k}).$$
We stratify the second coordinate of (19) by taking as the $j$’th stratum the $\mathbb{Z}/2$-quotient of the subspace of 

$$\mathbb{A}^{n+k} \times \mathbb{A}^{n+k}$$

consisting of pairs of elements whose all respective coordinates coincide except the first $j$. Assuming the characteristic of the ground field is not 2, the bottom pure stratum is

(20) $$\mathbb{A}^{n+k},$$

the higher strata are

(21) $$\mathbb{G}_m \times \mathbb{A}^{n+k+i},$$

leading to strata of (19) of type (13), as claimed. \hfill \Box

4. PROOFS OF THE THEOREMS AND THE MOTIVIC ADAMS-NOVIKOV SPECTRAL SEQUENCE

Although the statement of Proposition 9 is precisely as needed for the induction, note that the proof actually precisely accounts for the strata of type (13), showing that the motivic $\mathbb{Z}/2$-cohomology

(22) $$H^*_{\text{Mot}}(X', \mathbb{Z}/2)$$

in the stable range is the subspace of $\mathbb{Z}/2$-valued étale cohomology which is the tensor product over $\mathbb{Z}/2$ of $\mathbb{Z}/2[\theta]$ with the $\mathbb{Z}/2$-module with basis

(23) $$Q_k \ldots Q_1 \alpha$$

where $\alpha$ is the characteristic class of dimension $n(1 + \alpha)$, and $Q_i$ is either of the form $P^s$ or $\beta P^s$ where $P^s$ is of dimension $s(1 + \alpha)$, and $\beta$ is of dimension 1. Now recall Proposition 8. Passing from (23) to the cohomology of $X$ amounts to taking a certain direct summand. Passing to étale cohomology amounts to inverting $\theta$, but also in the étale cohomology one knows that the cohomology is the same as in the topological situation, tensored with $\mathbb{Z}/2[\theta, \theta^{-1}]$. Thus, in étale cohomology, the direct summand is obtained by imposing the Adem relations. But the Adem relations respect twist, and so we see that the summand of (22) corresponding to the cohomology of $X$ (in the stable range) is generated, as a $\mathbb{Z}/2[\theta]$-module, by admissible words of the form (23). Dimensional accounting ([9]) shows that the elements $P^s$, $\beta P^s$ we constructed must, in fact, be the reduced power operations and their Bocksteins. Thus, we have proved Theorem 2, and hence Corollary 3.

Let us now turn to Theorem 4. By Theorem 2 and [6], we know that the motivic Adams spectral sequence in our situation converges to the homotopy of $\text{MGL}$ (consider finite Thom spectra and pass to direct limit - convergence follows from the fact that the homotopy will eventually stabilize, i.e. remain constant, in dimensions $k + \ell \alpha$ with $k + \ell < N$, $N$ increasing).

Now (recall that we are over an algebraically closed field) one has

(24) $$H_{\mathbb{Z}/2}^* MGL = \mathbb{Z}/2[\xi_1, \xi_2, \ldots] \otimes \mathbb{Z}/2[m_i | i \neq 2^k - 1]$$

as a comodule over $H_{\mathbb{Z}/2}^* MGL$/$\mathbb{Z}/2^* Mot$ ($m_i$ are primitive, $i \neq 2^k - 1$), by arguments which parallel exactly the topological case. Now using the Adams spectral sequence, Theorem 4 follows.
Therefore, over an algebraically closed field $k$ of characteristic 0, we have formal group law theory for algebraically oriented spectra, which parallels the topological case (see also [3, 4, 5]). In particular, we have the Quillen idempotent, and $\text{MGL}^\wedge$ is a wedge of suspensions of copies of a spectrum $\text{BPGL}$ where

$$\text{BPGL}_* = \mathbb{Z}_2[\theta, v_1, v_2, \ldots], \quad \text{dim}(v_k) = (2^k - 1)(1 + \alpha).$$

Following arguments of Adams [1], we then see that the motivic analogue of the Adams-Novikov spectral sequence also converges to 2-completed stable homotopy groups. One also has an isomorphism of Hopf algebroids

$$(\text{BPGL}_*, \text{BPGL}_* \text{BPGL}) \cong (\text{BP}_*, \text{BP}_* \text{BP}) \otimes_{\mathbb{Z}_2} \mathbb{Z}_2[\theta]$$

where the Tate twist is primitive. Therefore, an analogous relation is true for the ANSS $E_2$-terms:

$$\text{Ext}^{s,t}_{\text{BPGL}_*}(1 + \alpha)^n(1 - \alpha)$$(25)

for $n \geq 0$ for $n < 0$.

Now to prove Theorem 1, we need a device for comparing differentials to the topological case. In characteristic 0, we know that maps of algebraically closed fields induce isomorphisms of 2-completed (not rational!) stable homotopy theory, and in the case of $k = \mathbb{C}$, we have a topological realization map, sufficient for our purposes. In the case of positive characteristic (for which, by the remarks of [9], our computations so far apply in a dimensional range $k + \ell \alpha, k + \ell < N$ for $N$ increasing with the prime), an alternate comparison device is the functor $G$ right adjoint to the pushforward from spectra to $\mathbb{A}^1$-spectra. Then $G^\text{et}$ is necessarily a ring spectrum, so we obtain a map $S \to G^\text{et}$. Since we can compute the behavior of this map in dimension 0, we can show by calculation that it is an equivalence.

Now in topology, the first non-trivial differential in the ANSS at $p = 2$ is

$$d_3 : \text{Ext}^2_{\text{BP}}^*, 28 \to \text{Ext}^5_{\text{BP}}^*, 30.$$ (26)

By the comparison, then, in the motivic case over an algebraically closed field, we must have a non-trivial

$$d_3 : \text{Ext}^2_{\text{BPGL}}^*, 14(1 + \alpha) \to \text{Ext}^5_{\text{BPGL}}^*, 14(1 + \alpha) + 2.$$ (27)

We see that the target of (27) is $\theta$ times the generator of $\text{Ext}^5_{\text{BPGL}}^*, 15(1 + \alpha)$ corresponding to the target $\alpha^2 \eta_{3/2}$ of (26). The examples in the $\alpha$ family are treated analogously, using the differentials originating in $\alpha_{4k+2}, \alpha_{4k+3}$. This concludes the proof of Theorem 1.

REFERENCES


