Math 593: Problem Set 3

Due Friday September 26, 2014

1. Determine all ideals in the ring \( \mathbb{Z}[x, y]/(2, x^2 + 1, y^2) \). Which are prime? Which are maximal?

2. Which of the following ideals are prime? Justify. Please make full use of the isomorphism theorems!
   a). Inside the ring of smooth real-valued functions on a manifold \( X \), the ideal of functions vanishing at a fixed \( p \in X \).
   b). The zero ideal in the ring of continuous functions on the interval \([0, 1]\).
   c). The ideal generated by \((3)\) in \( \mathbb{Z}[x, y]/(x^3 + 3x - 1) \).
   d). The ideal generated by \( i \) in the Gaussian integers \( \mathbb{Z}[i] \).
   e). The ideal generated by \((y, x^2 + yx + 1)\) in \( \mathbb{Z}[x, y] \).
   f). In the quotient ring \( \mathbb{R}[x, y]/(xy) \), the ideal generated by the class of \( x \).
   g). The ideal generated by \( \overline{x}, \overline{y} \) in \( \mathbb{F}_7[x, y, z, w]/(xy - zw + x^7 + xyz^8 + x^3 y^4 z^5 w^4) \).
   h) The ideal generated by \( 3y^{19} + x^{17} + 24x^{12}y + 9x^7y^2 + 12x^4y - 33x + 30 \) in \( \mathbb{Q}[x, y] \).
   i). The zero ideal in \( \mathbb{C}[x_1, \ldots, x_d]/(x_1^n + x_2^n + \ldots + x_d^n) \), where \( d \geq 3 \). [Hint: Induce on \( d \). Also, note that a homogeneous polynomial in two variables over \( \mathbb{C} \) factors completely into linear factors.]
   j). The ideal generated by \( y^2 - x^3 - x^2 \) in \( \mathbb{C}[x, y] \).
   k). The ideal generated by \( y^2 - x^3 - x^2 \) in the formal power series ring \( \mathbb{C}[[x, y]] \). [Hint: Think about taylor series expansions of \( \sqrt{x + 1} \).]

3. The Frobenius Map. a). Show that the integers are an initial object in the category of rings. [This means that for any ring \( R \), there exists a uniquely defined ring homomorphism \( \mathbb{Z} \to R \).]
   b). By definition, the characteristic of a ring \( R \) is the unique non-negative integer generating the kernel of the unique map you found in a. Show that the characteristic of any integral domain is either zero, or a prime number. Give examples to show that a ring which has infinite cardinality can have any integer characteristic \( n \geq 0 \).
   c). Let \( p > 0 \) be a prime integer. For any commutative ring \( R \) of characteristic \( p \), show that the map \( R \to R \) sending \( r \mapsto r^p \) is a ring homomorphism. This is called the Frobenius map. Can you find a non-zero ring \( R \) of characteristic zero in which the \( n \)-th power map is a ring
homomorphism for some \( n \geq 2 \)?

d). Show the Frobenius map on a commutative ring is injective if and only if the ring \( R \) has no nonzero nilpotent\(^1\) elements.

e). Find an example of a commutative ring for which the Frobenius map is surjective but not injective, and another for which it is injective but not surjective.

f). Does the category of rings admit a final object? (That is, is there some ring to which every ring can be mapped in one and only one way). Explain.

4. **Inverse Limits in the Category of Rings.** Do Exercise 10 on inverse limits from Dummit and Foote, Chapter 7, Section 6. Please only write down the details you yourself need to understand and check the results. Then do the following:

1. Let the indexing set \( I \) be the natural numbers, and let \( A_i = k[x]/(x^i) \), with the maps \( A_i \to A_j \) being the natural quotient maps \( k[x]/(x^i) \to k[x]/(x^j) \) whenever \( j \leq i \). Show that this is an inverse system of rings and find the limit \( \mathcal{A} \).

2. Use the universal property to find a natural map \( k[x] \to \mathcal{A} \). Describe this map. Is it injective?

5. A directed limit system of objects in any category is a collection of objects indexed by some directed partially ordered set (Cf. Problem Set 2), together with morphisms \( U_i \to U_j \) whenever \( j \geq i \) (respectively, \( j \leq i \) for inverse limit systems). Last week you defined the directed limit in the category of abelian groups, as well as in the category of commutative rings; you also showed direct limits always exists in these two categories.

a). Do direct limits exist in the category of Noetherian rings? Why or why not?

b). Fix a topological space \( X \). Let \( Op(X) \) be the category whose objects are the open sets of \( X \) and whose morphisms are the inclusions. Do direct limits always exist in \( Op(X) \)? Explain. Interpret the universal property of direct limits in this context: what obvious statement is equivalent?

c). What if we instead consider the category of closed sets \( Cl(X) \) in \( X \) (with morphisms inclusions)?

d). You might want to think about generalizing the notions of an inverse limits to any category as well...this makes for good office discussion but this problem set is long enough for now.

6. Do Exercise 11 on the \( p \)-adic numbers from Dummit and Foote, Chapter 7, Section 6.

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\(^1\)An element is nilpotent if some power is zero.