Math 593: Problem Set 7

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1 Basics properties of tensor product

i. We have a bilinear map \( M \times N \to N \otimes M \) given by \((m, n) \mapsto n \otimes m\) — by construction of the tensor product, \((m, n_1 + n_2) \mapsto n_2 \otimes m + n_1 \otimes m\), \((m, rn) \mapsto r(n \otimes m)\), \((m_1 + m_2, n) \mapsto n \otimes m_1 + n \otimes m_2\) and \((rm, n) \mapsto r(n \otimes m)\).

Then, by the universal property of tensor, we have a unique \( R \)-module map \( \varphi : M \otimes N \to N \otimes M \) s.t.

\[
\begin{array}{ccc}
M \times N & \longrightarrow & M \otimes N \\
\downarrow \quad \varphi & & \downarrow \varphi \\
N \otimes M & \longrightarrow & N \otimes M
\end{array}
\]

such that \( \varphi \) is given by \( n \otimes m \mapsto m \otimes n \). In the same way, we can construct an \( R \)-linear map \( \phi : N \otimes M \to M \otimes N \) which is inverse to \( \varphi \). Hence \( M \otimes N \cong N \otimes M \).

ii. For any fixed \( l \in L \), we have a bilinear map \( f_l : M \times N \to M \otimes (N \otimes L) \) defined by \((m, n) \mapsto m \otimes (n \otimes l)\). By the universal property of tensor products, we have a well-defined \( R \)-linear map \( f_l : M \otimes N \to M \otimes (N \otimes L) \). It is then easy to check that the map \((M \otimes N) \times L \to M \otimes (N \otimes L)\) given by \((m \otimes n, l) \mapsto f_l(m \otimes n) = m \otimes (n \otimes l)\) is well-defined and bilinear.

Then, applying again the universal property of tensor products, we have a \( R \)-module map \( \varphi : (M \otimes N) \otimes L \to M \otimes (N \otimes L) \) s.t.

\[
\begin{array}{ccc}
(M \otimes N) \times L & \longrightarrow & (M \otimes N) \otimes L \\
\downarrow \quad \varphi & & \downarrow \varphi \\
M \otimes (N \otimes L) & \longrightarrow & M \otimes (N \otimes L)
\end{array}
\]

On simple tensors, we have \( \varphi((m \otimes n) \otimes l) = m \otimes (n \otimes l) \).

Similarly, we get an \( R \)-module map \( \phi \) in the other direction, by first fixing an \( m \in M \) and defining a bilinear map \( N \times L \to (M \otimes N) \otimes L \) by \((n, l) \mapsto (m \otimes n) \otimes l\). We get an analogous an \( R \)-module map \( \phi : M \otimes (N \otimes L) \to (M \otimes N) \otimes L \) satisfying \( \phi(m \otimes (n \otimes l) = m \otimes (n \otimes l) \). So \( \phi \) and \( \varphi \) are inverses and we conclude that \((M \otimes N) \otimes L \cong M \otimes (N \otimes L)\).

iii. We have a bilinear map \((M \otimes N) \times L \to (M \otimes L) \otimes (N \otimes L)\) given by \((m \otimes n, l) \mapsto (m \otimes l, n \otimes l)\) — by construction of the tensor product and the properties of the direct sum, \((m_1 + m_2 \otimes n_1 + n_2, l) \mapsto ((m_1 + m_2) \otimes l, (n_1 + n_2) \otimes l) = (m_1, n_1) \otimes l + (m_2, n_2) \otimes l\), \((m \otimes n, l_1 + l_2) \mapsto (m \otimes (l_1 + l_2), n \otimes (l_1 + l_2)) = (m \otimes l_1, n \otimes l_1) + (m \otimes l_2, n \otimes l_2)\), \((m \otimes n, rl) \mapsto (m \otimes rl, n \otimes rl) = r(m \otimes l, n \otimes l)\).
Then, by the universal property of the tensor product, we have a unique $R$-module map $\varphi : M \otimes N \to N \otimes M$ s.t.

\[
\begin{array}{ccc}
(M \oplus N) \times L & \longrightarrow & (M \oplus N) \otimes L \\
\downarrow \varphi & & \downarrow \\
(M \otimes L) \oplus (N \otimes L)
\end{array}
\]

and we may verify by following the arrows that $\varphi$ is given by $(m, n) \otimes l \mapsto (m \otimes l, n \otimes l)$. We may verify that such a $\varphi$ is bijective on the simple tensors—it is injective since $m \otimes l = n \otimes l = 0$ implies $m + n \otimes l = 0$; it is surjective because we may hit any $m \otimes l$ by starting with $(m, 0) \otimes l$, and any $n \otimes l$ by starting with $(0, n) \otimes l$—and is hence a bijective $R$-module map between the tensor products.

Hence $(M \oplus N) \otimes L \cong (M \otimes L) \oplus (N \otimes L)$.

iv. Let $\mu : M \to M'$ and $\nu : N \to N'$ be isomorphisms. We have a bilinear map $M \times N \to M' \otimes N'$ given by $(m, n) \mapsto \mu(m) \otimes \nu(n)$—by construction of the tensor product and since $\mu$ and $\nu$ are $R$-module morphisms, $(m, n_1 + n_2) \mapsto \mu(m) \otimes \nu(n_1) + \mu(m) \otimes \nu(n_2)$, $(m, rn) \mapsto r(\mu(m) \otimes \nu(n))$, $(m_1 + m_2, n) \mapsto \mu(m_1) \otimes \nu(n) + \mu(m_2) \otimes \nu(n)$ and $(rn, n) \mapsto r(\mu(m) \otimes \nu(n))$.

Then, by the universal property of the tensor product, we have a unique $R$-module map $\varphi : M \otimes N \to M' \otimes N'$ s.t.

\[
\begin{array}{ccc}
M \times N & \longrightarrow & M \otimes N \\
\downarrow \varphi & & \downarrow \\
M' \otimes N'
\end{array}
\]

and we may verify by following the arrows that $\varphi$ is given by $m \otimes n \mapsto \mu(m) \otimes \nu(n)$.

Analogously, by working with the maps $\mu^{-1} : M' \to M$ and $\nu^{-1} : N' \to N$, we construct the inverse map. Hence $M \otimes N \cong M' \otimes N'$ whenever $M \cong M'$ and $N' \cong N'$ (as $R$-modules.)

## 2 Change of rings

Since $A \to B \to C$, every $C$ module is a $B$-module, and every $B$-module is an $A$-module. We abuse notation by writing $b$ for both $b \in B$ and its image in $C$, and likewise $a$ will denote $a \in A$ or its image in $B$ or its image in $C$. In all cases, the meaning should be clear from context.

There is a $B$-bilinear map $C \times (B \otimes_A M) \to C \otimes_A M$ given by $(c, b \otimes m) \mapsto cb \otimes m$ (where [in the left tensor factor] on the RHS $b$ denotes the image thereof in $C$ by the abuse mentioned above). This map is well-defined by the linearity properties of the tensor product. Now by the universal property of the tensor product, we can find a unique $B$-module map $\varphi : C \otimes_B (B \otimes_A M) \to C \otimes M$ s.t.

\[
\begin{array}{ccc}
C \times (B \otimes_A M) & \longrightarrow & C \otimes_B (B \otimes_A M) \\
\downarrow \varphi & & \downarrow \\
C \otimes_A M
\end{array}
\]

In fact, it is easy to see that this map is indeed $C$-linear, since $c\varphi(c' \otimes (b \otimes m)) = c(c'b \otimes m) = cc'b \otimes m = \varphi(c(c' \otimes (b \otimes m)))$.

There is also an $A$-bilinear map $C \times M \to C \otimes_B (B \otimes_A M)$ given by $(c, m) \mapsto c \otimes (1 \otimes m)$

We verify this is bilinear by checking that
• \((rc_1 + c_2, m) \mapsto (rc_1 + c_2) \otimes (1 \otimes m) = r(c_1 \otimes (1 \otimes m)) + c_2 \otimes (1 \otimes m)\)

• \((c, rm_1 + m_2) \mapsto c \otimes (1 \otimes (rm_1 + m_2)) = r(c \otimes (1 \otimes m_1)) + c \otimes (1 \otimes m_2)\)

Now by the universal property of the tensor product, we can find a unique \(A\)-module map \(\psi : C \otimes_A M \to C \otimes_B (B \otimes_A M)\) s.t.

\[
\begin{array}{c}
C \times M \\
\downarrow \psi \\
C \otimes_B (B \otimes_A M)
\end{array}
\]

**Claim.** \(\varphi\) and \(\psi\) are inverses, so in particular they are bijective.

To see this, note that \((\psi \circ \varphi)(c \otimes (b \otimes m)) = \psi(cb \otimes m) = cb \otimes (1 \otimes m) = c \otimes (b \otimes m)\) and \((\varphi \circ \psi)(cb \otimes m) = cb \otimes (1 \otimes m) = cb \otimes m\).

Hence \(C \otimes_B (B \otimes_A M) \cong C \otimes_A M\).

### 3 Tensor products in the category of \(A\)-algebras

(a) First we show that \(R \otimes_A S\) has a ring structure. \(R \otimes_A S\), being an \(A\)-module, is already an abelian group under \(+\). Now define multiplication on \(R \otimes_A S\) by \((r \otimes s) \cdot (r' \otimes s') = rr' \otimes ss'\) extended linearly. This is well-defined by the following argument: the universal property of the tensor product gives us well-defined \(A\)-linear maps \(R \otimes_A R \to R\) and \(S \otimes_A S \to S\), from

\[
\begin{array}{c}
R \times R \\
\downarrow (r,r') \\
R
\end{array}
\quad \quad \quad
\begin{array}{c}
S \times S \\
\downarrow (s,s') \\
S
\end{array}
\]

and then, again by the universal property of the tensor product, we have an \(A\)-linear map \(\varphi : (R \otimes_A S) \otimes_A (R \otimes_A S) \to R \otimes_A S\) s.t.

\[
\begin{array}{c}
(R \otimes_A R) \times (S \otimes_A S) \\
\downarrow (r \otimes r', s \otimes s') \\
R \otimes_A S
\end{array}
\]

and now we may compose the natural map \((R \otimes_A S) \times (R \otimes_A S) \to (R \otimes_A S) \otimes (R \otimes_A S)\) with \(\varphi\) to obtain the above-described map \((R \otimes_A S) \times (R \otimes_A S) \to R \otimes_A S\).

We note that multiplication hence defined is associative, by the associativity of multiplication in \(R\) and \(S\), and has \(1 \otimes 1\) as an identity element. Moreover multiplication distributes across addition by bilinearity.

Now define an \(A\)-algebra structure on \(R \otimes_A S\) by \(a \mapsto a(1 \otimes 1)\). This is a ring homomorphism because \((a + b) \mapsto (a + b)(1 \otimes 1) = a(1 \otimes 1) + b(1 \otimes 1)\), \(ab \mapsto ab(1 \otimes 1) = a(b(1 \otimes 1))\), \(1 \mapsto (1 \otimes 1)\), and \(0 \mapsto 0(1 \otimes 1) = 0\).

Moreover the image of \(A\) is clearly in the center of \(B\) (with multiplication as given above.)

(b) Define a map \(R \to R \otimes_A S\) by \(r \mapsto r \otimes 1\) and a map \(S \to R \otimes_A S\) by \(s \mapsto 1 \otimes s\). These are ring maps by the way we have defined ring operations on \(R \otimes_A S\). To see that they are also \(A\)-algebra maps, we need a commutating diagram
but this is easy since $A \to R \to R \otimes_A S$ sends $a \mapsto a \otimes 1 = a(1 \otimes 1)$. Similarly with $S \to R \to S$ sending $s \mapsto 1 \otimes s$ is an $A$-algebra map.

Now suppose we have a commutative $A$-algebra $T$ s.t. we have $A$-algebra maps $R \to T$ and $S \to T$. We wish to show that we can find a unique $A$-algebra map $\varphi : R \otimes_A S \to T$ s.t.

so that $R \otimes_A S$ is indeed a coproduct (of $R$ and $S$) in the category of commutative $A$-algebras.

Since $R \times S \to T$ sending $(r, s) \mapsto rs$ is $A$-bilinear, the universal property of tensor products gives

We may verify that $\varphi$ is a ring map: $\varphi(1 \otimes 1) = 1$ and $\varphi(0 \otimes 1) = 0$, and $\varphi$ respects ring addition by definition, and $\varphi((r \otimes s)(r' \otimes s')) = rr's's' = (rs)(r's')$. Moreover $\varphi$ is an $A$-algebra map, since $\varphi(a(1 \otimes 1)) = a \in T$.

(c) Commutative rings are $\mathbb{Z}$-algebras, so the result of the previous part applied to $A = \mathbb{Z}$ shows that coproducts exist in the category of commutative rings: in particular, given two rings $R$ and $S$, the tensor product $R \otimes \mathbb{Z} S$ is a coproduct of $R$ and $S$ in the category of commutative rings.

4 $A$-algebra isomorphisms

a. Define the map $\varphi : B \otimes_A A[x, y] \to B[x, y]$ by $(b \otimes f) \mapsto bf$. This is an isomorphism, from the chain of isomorphisms

$$
B \otimes_A A[x, y] \cong B \otimes_A \left( \bigoplus_{i,j \in \mathbb{Z}_{\geq 0}} A(x^i y^j) \right) \\
\cong \bigoplus_{i,j} (B \otimes_A A)(x^i y^j) \\
\cong \bigoplus_{i,j} B(x^i y^j) \cong B[x, y]
$$

b. We start with the exact sequence

$$
0 \to (f_1, \ldots, f_t) \longrightarrow R[x_1, \ldots, x_d] \longrightarrow R[x_1, \ldots, x_d]/(f_1, \ldots, f_t) \longrightarrow 0
$$
and now we apply the functor $R/I \otimes_R -$ to our sequence

We know from [a similar argument to] the above, with $A = R$ and $B = R/I$, that $R/I \otimes_R R[x_1, \ldots, x_d] \cong R/I[x_1, \ldots, x_d]$. Similarly, $R/I \otimes_R (f_1, \ldots, f_t) \cong (f_1, \ldots, f_t)$.

By right-exactness of $\otimes$ (and implicitly appealing to the above isomorphisms where required), we obtain the exact sequence

$$\text{im}(R/I \otimes_R (f_1, \ldots, f_t)) = (\bar{f}_1, \ldots, \bar{f}_t) \longrightarrow R/I[x_1, \ldots, x_d] \longrightarrow R/I \otimes_R R[x_1, \ldots, x_d]/(f_1, \ldots, f_t) \longrightarrow 0$$

whence we may conclude $R/I \otimes_R R[x_1, \ldots, x_d]/(f_1, \ldots, f_t) \cong R/I[x_1, \ldots, x_d]/(\bar{f}_1, \ldots, \bar{f}_t)$.

c. This follows from the result of the previous part with $R = \mathbb{Z}$, $I = (5)$, $d = 2$, $t = 1$ and $f_1(x, y) = x^5 + 5xy + y^5$, where we verify that $(x + y)^5 = x^5 + 5xy + y^5$ when we reduce all coefficients modulo $I = (5)$.

5 True or false

- True. We may verify that the map $\mathbb{R} \rightarrow \mathbb{R} \otimes_\mathbb{Z} \mathbb{Q}$ given by $r \mapsto (r \otimes 1)$ is an isomorphism (of $\mathbb{Z}$-modules, i.e. of abelian groups): it is certainly injective, and it is surjective since any $s \otimes \frac{a}{b} = as \otimes \frac{1}{b} = \frac{as}{b} \otimes 1$.

- False. The LHS (as a $\mathbb{R}$-vector space) is generated by the basis $\{1 \otimes 1, 1 \otimes i, i \otimes 1, i \otimes i\}$, and so has dimension 4, but the LHS has dimension 2 (being generated by 1 and i.)

- False, the LHS (as a $\mathbb{R}$-vector space) has basis $\{i \otimes 1, 1 \otimes 1\}$ (and, in particular, is two-dimensional), but the RHS is a one-dimensional $\mathbb{R}$-vector space.

- True, $z \otimes w = wz \otimes 1$, so $C \otimes_C C \cong C$ (as $C$-modules) by the $C$-vector space isomorphism $z \otimes w \mapsto wz$.

- True. $k[x] \otimes_k k[y]$ is a $k$-vector space with basis $(x^a \otimes y^b)_{a,b \in \mathbb{Z}_{\geq 0}}$. $k[x, y]$ is a $k$-vector space with basis $(x^a y^b)_{a,b \in \mathbb{Z}_{\geq 0}}$. Now we can put the bases in bijection with other by sending $x^a \otimes y^b \mapsto x^a y^b$; hence the two are isomorphic (as $k$-modules / $k$-vector spaces.)

6

First note $\mathbb{Z}[x, y]/(x^3 \otimes \mathbb{Z}[x, y]) \cong \mathbb{Z}[x, y]/(x^3, y^2)$; then observe that $\mathbb{Z}_{10} \otimes_\mathbb{Z} \mathbb{Z}[x, y]/(x^3, y^2) \cong \mathbb{Z}[x, y]/(10, x^3, y^2)$.

Now the ideals of $\mathbb{Z}[x, y]/(10, x^3, y^2)$ are in one-one correspondence with the ideals of $\mathbb{Z}[x, y]$ containing $(10, x^3, y^2)$; under this correspondence, primes correspond to primes (by the third isomorphism theorem and the fact that $P$ is prime if and only if $R/P$ is a domain).

But a prime $P$ containing $y^2 = y \cdots y$ must contain $y$; likewise if $x^3 = x \cdots x^2$ must contain $x$. Also, since $10 \in P$, either 2 or 5 is in $P$. So the only options for $P$ are $(2, x, y)$ and $(5, x, y)$. Thus our ring has exactly 2 prime ideals.
(a) Let \( R \) denote our PID. We have
\[
M = R^{r_1} \oplus R/(f_1) \oplus \cdots \oplus R/(f_t)
\]
\[
N = R^{r_2} \oplus R/(g_1) \oplus \cdots \oplus R/(g_s)
\]
and now
\[
M \otimes_R N = (R^{r_1} \oplus R/(f_1) \oplus \cdots \oplus R/(f_t)) \otimes_R (R^{r_2} \oplus R/(g_1) \oplus \cdots \oplus R/(g_s))
\]
\[
= R^{r_1r_2} \oplus R^{r_1} \otimes R/(g_1) \oplus \cdots \oplus R^{r_1} \otimes R/(g_s) \oplus R/(f_1) \otimes R^{r_2} \oplus \cdots \oplus R/(f_t) \otimes R^{r_2}
\]
\[
= R^{r_1r_2} \bigoplus_{1 \leq i \leq t, 1 \leq j \leq s} (R/(f_i) \otimes_R R/(g_j))
\]
In particular, the rank is \( r_1r_2 \).

The last invariant factor is the annihilator of the torsion part of the module, which is displayed as a direct sum of cyclic modules. Recall from a previous problem set that the annihilator of a direct sum is the intersection of their annihilators. So we need to find the intersection of all the ideals \( (f_i), (g_j) \) if \( r_2 \), respectively \( r_1 \) is not zero and \( (f_i, g_j) = \gcd(f_i, g_j) \). There are many inclusions among these ideals, since \( (f_1) \supset (f_2) \supset \cdots \supset (f_t) \) and \( (g_1) \supset (g_2) \supset \cdots \supset (g_s) \).

Indeed, we only need to consider the ideals \( (f_i), (g_s) \) and \( (f_t, g_s) \). Analyzing this,
- If \( r_1r_2 > 0 \), then the top invariant factor is \( \text{lcm}(f_i, g_s) \), since this generates \( (f_t) \cap (g_s) = (f_1) \cap (g_s) \).
- If \( r_1 \neq 0 \) but \( r_2 = 0 \), then the top invariant factor is \( g_s \), since \( (g_s) \cap (f_1, g_s) = (g_s) \).
- If \( r_2 \neq 0 \) but \( r_1 = 0 \), then the top invariant factor is \( f_t \), since \( (f_t) \cap (f_t, g_s) = (f_t) \).
- If \( r_1 = r_2 = 0 \), then the top invariant factor is \( \gcd(f_t, g_s) \), (provided it is not a unit!) since this generates \( (f_t, g_s) \), which annihilates all \( R/(f_1, g_j) \). 

The first invariant factor will be moved to another problem set, with substantial hints. Part b will also appear again.

8 A QR problem

Let \( Mat_{2 \times 2}(\mathbb{C}) \) be the space of \( 2 \times 2 \) matrices with complex entries. Fix an element \( A \in Mat_{2 \times 2}(\mathbb{C}) \).

Show that the self-map \( L \) sending \( X \) to \( AX -XA \) is a \( \mathbb{C} \)-vector space endomorphism. When \( A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \), find its Jordan form.

Let \( L: Mat_{2 \times 2}(\mathbb{C}) \to Mat_{2 \times 2}(\mathbb{C}) \) be as above. First we show that \( L \) is a \( \mathbb{C} \) linear transformation. This is just a restatement of the fact that matrix multiplication distributes over matrix addition and that scalar multiplication (by \( \lambda \in \mathbb{C} \)) commutes with matrix multiplication. Symbolically
\[
(A + B)C = AC + BC, \quad \text{and} \quad C'(A + B) = C'A + C'B, \quad \text{and} \quad A(\lambda B) = \lambda AB.
\]
Now, fix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and fix a basis for $\text{Mat}_{2 \times 2}(\mathbb{C})$ as follows

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. $$

It should be clear that these matrices are linearly independent over $\mathbb{C}$ and that they span all of $\text{Mat}_{2 \times 2}(\mathbb{C})$. We find how $L$ acts on each of these basis vectors.

$L(e_1) = Ae_1 - e_1A = e_1^2 - e_1^2 = 0,$ since $e_1 = A.$

$L(e_2) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = e_2$

$L(e_3) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = -e_3$

$L(e_4) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0$

This means that in this ordered basis $B = \{e_1, e_2, e_3, e_4\}$ the transformation $L$ is presented by the matrix

$$[L]_B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

which is already in Jordan form.

9 Simple tensors

(a) Given $V$ and $W$ of dimensions $d$ and $e$, we fix bases so that they are identified with $k^d$ and $k^e$, respectively. We interpret $V$ as the column space $k^d$ and $W$ as the row space $k^e$. There is a bilinear map $k^d \times k^e \rightarrow \text{Mat}_{de}(k)$ given by matrix multiplication:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix}, \begin{bmatrix} b_1 & b_2 & \ldots & b_e \end{bmatrix} \mapsto \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_e \end{bmatrix}.$$ 

This induces a linear map $V \otimes_k W \rightarrow M_{de}(k)$ which sends the basis elements $v_i \otimes w_j$ to the basis elements $E_{ij}$ of $\text{Mat}_{de}(k)$, the $d \times e$ matrix with a 1 in the $de$ entry and zeros everywhere else. Because this linear map is a bijection on basis elements, it is an isomorphism.

To understand the simple tensor $v \otimes w$ in this set-up, note that $v$ is interpreted as a row and $w$ as a column. Their product is rank one; indeed, the $j$-column is $b_j$ times the column $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix}$, so all columns are scalar multiples of each other. Conversely, a rank one matrix corresponds to a
(non-zero) simple tensor: take any non-zero column, say the \( j \)-th column \[
\begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_d
\end{pmatrix}
\]. Every other column is a scalar multiple of this one, say column \( i \) is \( b_i \) times column \( j \) (note: \( b_j = 1 \)). So the rank one matrix is \((a_i b_j)\), which means that it is the product
\[
\begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_d
\end{pmatrix} \begin{bmatrix} b_1 & b_2 & \ldots & b_e \end{bmatrix}.
\]
This means it corresponds to the simple tensor \( v \otimes w \) where \( v = \sum a_i v_i \) and \( w = \sum b_j w_j \).

To count the number of rank one matrices, we use the fact that \((k^d \setminus 0) \times (k^e \setminus 0)\) maps onto the set of rank one matrices. This is not a bijection (kudos to Bob Lutz for being possibly the only 593 to realize this). Indeed, for any non-zero scalar \( \lambda \), the pairs \((v, w)\) and \((\lambda v, \lambda^{-1} w)\) have the same image (after multiplying column \( v \) times row \( w \)). But this is the only way two matrices can multiply to the same matrix, so each rank one matrix has exactly \( q - 1 \) pre images. This means that the set of rank one matrices has cardinality \( \frac{(q^d - 1)(q^e - 1)}{q - 1} \), and so (adding in the zero tensor) there are exactly \( \frac{(q^d - 1)(q^e - 1)}{(q - 1)^2} \) simple tensors.

So the fraction of simple tensors is \( \frac{(q^d - 1)(q^e - 1)}{(q - 1)^2} \). This is clearly a small fraction, roughly \( \frac{q^d + q^e}{q} \), unless \( d \) or \( e \) is one, in which case all tensors are simple. In particular, it goes to zero as \( q \) gets large.

In \( \mathbb{R}^d \otimes \mathbb{R}^e \cong \mathbb{R}^{de} \), the set of simple tensors corresponds the pre image of zero under the map
\[
Mat_{de}(\mathbb{R}) \to \mathbb{R}^{(d \times e)(2)}
\]
sending each \( d \times e \) matrix to the list of determinants of all its \( 2 \times 2 \) subminors (note that a matrix has rank one or less if and only if all \( 2 \times 2 \) minors are zero). So it is a proper closed set (if \( e \) and \( d \) are greater than one) and by the inverse function theorem (removing 0) can be given the structure of a manifold of strictly smaller dimension. So is has Lebesgue measure zero in \( Mat_{de}(\mathbb{R}) \).

Put differently, if we randomly chose an element in \( V \otimes W \) (with \( d, e \geq 2 \)), the probability that it is a simple tensor is zero.