Assume the ground field \(k\) is algebraically closed, unless stated otherwise.

1. a). Prove that every regular map of algebraic sets is continuous in the Zariski topology. Find a counterexample to the converse.

2. Let \(F : V \to W\) be a regular mapping of algebraic sets and let \(F^* : k[W] \to k[V]\) be the induced map of coordinate rings.
   a). Prove that \(F\) is dominant if and only if \(F^*\) is injective. (A map is dominant if its image is dense in the target space.)
   b) Prove that \(F\) is a closed embedding if and only if \(F^*\) is surjective. (A closed embedding is an isomorphism onto a closed set of the target space.)

3. a). At what points of the “circle” given by \(V(x^2 + y^2 - 1)\) in \(\mathbb{A}^2\) is the rational function \((1 - y)/x\) regular? (Your answer might depend on the characteristic.)
   b). At what points of the “nodal curve” given by \(V(y^2 - x^2 - x^3)\) in \(\mathbb{A}^2\) is the rational function \(y/x\) regular?
   c). Prove that if \(V\) is an irreducible algebraic set and \(f/g \in k(V)\) satisfies \(g(p) = 0\) and \(f(p) \neq 0\), then \(f/g\) is not regular at \(p\).
   c). Find the locus of points at which the rational function \(x/z\) is regular on the algebraic set \(V(xy - zw)\) in \(\mathbb{A}^4\).

4. Let \(V\) be an algebraic subset of \(\mathbb{A}^n\), and let \(k[V]\) be its coordinate ring.
   a). For each \(f \in k[V]\), let \(U_f\) be the set of points \(P \in V\) such that \(f(P) \neq 0\). Prove that the collection of all such sets \(U_f\) forms a basis for the Zariski topology on \(V\). The set \(U_f\) is called a basic (or principle) open set.
   b). Prove that \(V\) is compact (every open cover has a finite subcover).
   c). Is \(V\) Hausdorff? Describe all irreducible Hausdorff algebraic sets.

5. With notation as in Exercise 4, consider the map
   \[
   \phi : U_f \to \mathbb{A}^n \times \mathbb{A}^1
   \]
   \[
   P \mapsto (P, \frac{1}{f(P)}).
   \]
   a). Show that this map defines a homeomorphism from \(U_f\) onto an algebraic subset \(W\) of \(\mathbb{A}^{n+1} = \mathbb{A}^n \times \mathbb{A}^1\). (The topology on \(U_f\) is the subspace topology induced by the Zariski topology on \(V\).)
   b) Find the coordinate ring of \(W\) and show that it is isomorphic to \(k[V][\frac{1}{f}]\).
   c). Assuming \(V\) is irreducible, find the ring of regular functions \(O_V(U_f)\) on the open set \(U_f\), and show that the pullback of the map \(\phi\) induces an isomorphism between the rings of regular functions on \(W\) and \(U_f\).
d). Conclude that (the Zariski topology on) $V$ has a basis of sets that can be identified with affine algebraic sets (in some possibly different ambient $k^n$).

6. The ringed space structure on an algebraic set.

a). Look up the definition of a presheaf on page 61 of Hartshorne. Let $X$ be a topological space. Prove that the assignment taking an open set $U$ of $X$ to the set of all $\mathbb{R}$-valued functions $U \to \mathbb{R}$ is a pre-sheaf of $\mathbb{R}$-algebras on $X$. Prove that similarly, the assignment taking an open set $U$ of $X$ to the subset of all $\mathbb{R}$-valued functions $U \to \mathbb{R}$ which are continuous (respectively bounded, respectively constant) is also a pre-sheaf of rings (or $\mathbb{R}$-algebras) on $X$. Assuming $X$ is a manifold, do the same for smooth functions. Assuming $X$ is an open subset of $\mathbb{C}$, do the same for holomorphic functions.

b). Determine which of the six pre-sheaves in part a satisfy the sheaf axiom: If $U = \bigcup_{i} U_{\lambda_i}$ is an open cover of $U$ and for each $\lambda$, $g_{\lambda}$ is section over $U_{\lambda_i}$ whose restrictions to all intersections $U_{\lambda_1} \cap U_{\lambda_2}$ agree, then there exists a unique section $g$ over $U$ whose restriction to $U_{\lambda_i}$ is $g_{\lambda}$.

c) Let $V$ be an irreducible algebraic set, with function field $k(V)$. For each open set $U \subset V$, let $\mathcal{O}_V(U)$ be the set of all rational functions $\phi \in k(V)$ that are regular at each point $P \in U$. Prove that the assignment $U \mapsto \mathcal{O}_V(U)$ makes $\mathcal{O}_V$ into a sheaf of $k$-algebras of functions on $V$.

d). A ringed space $(X, \mathcal{O}_X)$ is a topological space $X$ together with a sheaf of rings on it. Deduce that every irreducible algebraic set has the structure of an ringed space. Explain also how topological spaces with continuous functions, differentiable manifolds with smooth functions, and complex manifolds with holomorphic functions are naturally ringed spaces as well. In fact, all of these are all sheaves of algebras of functions on $X$, that is, subsheaves of the pre sheaf in part (a).

e). Show that for any open set $U$ of $X$, there is a naturally induced ringed space structure $(U, \mathcal{O}_{X|U})$ on $U$.

f). We say that two ringed spaces of functions $(V, \mathcal{O}_V)$ and $(X, \mathcal{O}_X)$ are isomorphic if there is a homeomorphism $\phi : V \to X$ which induces an isomorphism of rings (under pullback of functions) for each open set $U$ of $X$: $\phi^* : \mathcal{O}_X(U) \to \mathcal{O}_V(\phi^{-1}(U))$ sending $g \mapsto g \circ \phi$, compatible with restriction. Prove that (with notation as in Exercise 5), the map from $U_f$ to $W$ induces an isomorphism between ringed spaces $(U_f, \mathcal{O}_{V|U_f})$ and $(W, \mathcal{O}_W)$.

7. Abstract Varieties. An algebraic variety over $k$ can be defined as a ringed space $(V, \mathcal{O}_V)$ which admits an open cover $\{U_i\}_{i \in I}$ such that each $(U_i, \mathcal{O}_{V|U_i})$ is isomorphic to the ringed space of some irreducible algebraic set in some $k^n$ (with $n$ possibly depending on the choice of open set $U_i$). 3

a). Show that every open subset of an algebraic set has the structure of an abstract variety.

b). A abstract variety is said to be affine if it is isomorphic (as ringed space) to an algebraic subset of some $k^n$. Show that that if $V$ is an irreducible algebraic set, then any basic open set $U_f$ is an affine algebraic variety. We say that it is an open affine chart of $V$.

c). Show that $GL_n$ has the structure of an affine variety even though it is not an algebraic subset of $\mathbb{A}^n$. 2

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1It is only slightly more technical algebraically to figure out the case where $V$ has multiple components. You don’t have to do this here, but might want to think about it.

2or just $\mathbb{C}^n$ if you aren’t familiar with complex manifolds.

3Some authors would call this a prevariety, and require an additional assumption on the topological space called ”separatedness” (which amounts to something like the Hausdorff axiom for manifolds) in order for it to be a variety. In fact, a variety over the complex numbers is separated if and only if it is Hausdorff when considered with the euclidean topology. Note that all complex varieties have a Euclidean topological since since they locally look like subsets of complex space. We won’t go into separatedness in Math 631, because the main examples we study will automatically satisfy this condition, as do essentially all the examples that come up for the algebraic geometer on the street.