1. Let $X$ be a variety smooth in codimension one. Let $Y \subset X$ be a prime divisor.
   a). Show that there is an open affine set $U \subset X$ such that $\mathcal{I}_Y(U) = (\pi)$ for some regular function $\pi \in \mathcal{O}_X(U)$, some $U$ which meets $Y$.
   b). Show that if $U' \subset U$, then $\pi_{|U'}$ generates $\mathcal{I}_Y(U')$.
   c). For $U$ affine as in (a), and $\phi$ regular on $U$, define $\nu^U_Y(\phi)$ as the largest integer $r$ such that $\phi \in (\pi^r)$. Show that this is independent of the choice of $U$. Call this $\nu_Y(\phi)$.
   d). Prove that the local ring $\mathcal{O}_{X,Y}$ of $X$ along the subvariety $Y$ is a discrete valuation ring.¹
   e). Show that $\nu_Y$ agrees with the valuation determined by the $\mathcal{O}_{X,Y}$.

2. Non-principal divisors on affine smooth varieties. Let $X = \mathbb{V}(y^2 - x^3 + x) \subset \mathbb{A}^2$, where the characteristic is not 2 or 3.
   a). Show that $X$ is smooth (hence all prime divisors are locally principal).
   b). Show that the origin $P$ is a prime divisor on $X$ which is not principal.²
   c). Find an explicit local defining equation for $P$ in a neighborhood of $P$.

3. Maps of Picard groups for curves. Let $X = \mathbb{V}(z^d - f(x, y)) \subset \mathbb{P}^2$ be a smooth curve, where $f(x, y)$ is a homogeneous polynomial of degree $d$. Consider projection $\pi : X \to \mathbb{P}^1$ sending $[x : y : z] \mapsto [x : y]$.
   a). Let $P$ be a point of $\mathbb{P}^1$. Compute $\pi^*P$ explicitly. Draw a sketch of this divisor on $X$, thinking of $X$ as a $d$-sheeted cover of $\mathbb{P}^1$.
   b). Prove that the pull back map $\text{Pic}(\mathbb{P}^1) \to \text{Pic}(X)$ sends $[D]$ to a divisor class of degree $(\deg \pi)(\deg D)$.
   c). More generally, for any degree $d$ map of smooth projective curves $X \to Y$, prove that the induced map on Picard groups multiplies degrees by $d$.

4. Normal Varieties. Reread Shafarevich on Normal varieties. Recall the 614 Theorem: If $R$ is a Noetherian domain, then $R$ is normal if and only if the natural inclusion $R \subset \bigcap_{P \text{ht} R} R_P$ (in the fraction field) is an equality.
   a). Be sure you understand why every smooth variety is normal (hint: use that the local rings of a smooth varieties are UFDs).
   b). Prove that if $X$ is a normal (for example, smooth) variety, then a rational function $\phi \in k(X)$ is regular if and only if it has no poles, that is, if $\text{div} \phi$ is effective.
   c). Let $X = \mathbb{V}(P) \subset \mathbb{A}^4$ where $P$ is the kernel of the “obvious” surjective algebra map $k[x, y, z, w] \to k[s^4, s^3 t, st^3, t^4]$. Prove that $X$ is smooth in codimension one.³ Is the rational function $\frac{y^2}{x}$ regular? Compute $\text{div} \frac{y^2}{x}$. Show that $X$ is not normal by showing $\frac{y^2}{x}$ is in the normalization. Does the statement in b hold for non-normal varieties?

¹Use whatever definition you are familiar with; one definition is a local Noetherian domain whose maximal ideal is principal, but make sure this definition agrees with whatever one you know.
²Actually, I believe no prime divisor in this ring is principal, but I won’t ask you to prove it.
³Hint: show all $U_{x_i}$ are smooth.
5. Pulling back divisors. For each map below, describe explicitly the pullback of hyperplane divisors. Eg: Is the pullback defined? When is it a prime divisor? What is the (bi)degree (if defined)? In "sample chart,” how does it look?

1. The Veronese map of degree d, \( v: \mathbb{P}^n \to \mathbb{P}^{(n+d)} \).
2. The Segre map \( \sigma: \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^{nm+n+m} \).
3. The Plucker embedding \( G(d, k^n) \to \mathbb{P}(\wedge^d k^n) \).
4. The blowup of \( \pi: X \to \mathbb{P}^2 \) at the origin. [Caution! It could depend on which line you pull back...how?]
5. The projection from a point \( p \) not on a hypersurface \( X \in \mathbb{P}^{d+1} \): \( \pi_p: X \to \mathbb{P}^d \).

6. Invertible Sheaves. Let \( X \) be an irreducible normal variety, \( D \) a divisor on \( X \). For every open set \( U \) of \( X \), define \( \mathcal{O}_X(D)(U) = \{ f \in k(U)^* \mid div(f) + (D \cap U) \geq 0 \} \cap 0 \).

a). Show that \( \mathcal{O}_X(D) \) defines a sheaf of abelian groups on \( X \).

b). Show that \( \mathcal{O}_X(D) \) has the structure of sheaf of \( \mathcal{O}_X \)-modules. [This means that each \( \mathcal{O}_X(D)(U) \) is an \( \mathcal{O}_X(U) \)-module, compatibly with restriction maps: if \( V \subset U, \phi \in \mathcal{O}_X(U) \) and \( g \in \mathcal{O}_X(D)(U) \), then the restriction of \( \phi g \) to \( V \) is equal to \( \phi|_V g|_V \).]

c). Show that if \( D \) is Cartier, then \( \mathcal{O}_X(D) \) is a locally free \( \mathcal{O}_X \)-module of rank one. [This means that \( X \) has a cover by open affine sets \( U \) such that each \( \mathcal{O}_X(D)(U) \cong \mathcal{O}_X(U) \) as modules over \( \mathcal{O}_X(U) \).

d). Let \( D = 4L - C \) be the divisor on \( \mathbb{P}^2 \) where \( L \) is the line \( \mathbb{V}(x) \) and \( C \) is the smooth cubic \( \mathbb{V}(x^3 + y^3 + z^3) \). Explicitly compute the sections of \( \mathcal{O}_X(D) \) over each of the standard affine charts, as well as the global sections of this sheaf.

7. The tautological Bundle. Let \( \pi: L \to \mathbb{P}^n \) be the tautological bundle on \( \mathbb{P}^n \) as defined in Problem set 5. So \( L \subset \mathbb{A}^{n+1} \times \mathbb{P}^n \) is the incidence correspondence consisting of \((x, \ell) \mid x \in \ell, \) and \( \pi \) is the projection to the second factor.

a). A section of the tautological bundle over \( U \subset \mathbb{P}^n \) is a regular map \( s: U \to L \) such that \( \pi \circ s = Id_U \). Prove that the set \( \mathcal{L}(U) \) of all sections over \( U \) is a \( \mathcal{O}_{\mathbb{P}^n}(U) \) module. Prove that \( \mathcal{L} \) forms a sheaf of \( \mathcal{O}_{\mathbb{P}^n} \)-modules.

b). For a standard affine chart \( U_i \), prove that \( \mathcal{L}(U_i) \) is a free \( \mathcal{O}_{\mathbb{P}^n}(U_i) \) module of rank one.\(^4\)

c). Show that \( \mathcal{L}(U_i \cap U_j) \) is the localization of the \( \mathcal{O}_{\mathbb{P}^n} \)-module \( \mathcal{L}(U_i) \) at the regular function \( x_j/x_i \in \mathcal{O}_{\mathbb{P}^n}(U_i) \).

d). Let \( \phi_i: \mathcal{O}_{\mathbb{P}^n}(U_i) \to \mathcal{L}(U_i) \) be an isomorphism as in b. Let \( \rho_{ij} \) be the restriction map \( \mathcal{L}(U_i) \to \mathcal{L}(U_i \cap U_j) \). Explain why both \( \rho_{ij}(\phi_i(1)) \) and \( \rho_{ij}(\phi_j(1)) \) are free generators for the rank one free \( \mathcal{O}_{\mathbb{P}^n}(U_i \cap U_j) \)-module \( \mathcal{L}(U_i \cap U_j) \). Explain why this allows us to interpret the fraction \( \rho_{ij}(\phi_i(1))/\rho_{ij}(\phi_j(1)) \) as an invertible element of \( \mathcal{O}_{\mathbb{P}^n}(U_i \cap U_j) \). Explicitly compute this invertible element for your isomorphisms found in b.

e). Prove that there are no non-zero global sections of the tautological bundle, that is, that \( \mathcal{L}(\mathbb{P}^n) = 0 \).

8. Complete Linear systems. Fix a divisor \( D \) on a smooth variety \( X \). Let \( |D| \) be the set of all effective divisors linearly equivalent to \( D \). Thus \( D' \in |D| \) if and only if there exists a \( \phi \in k(X) \) such that \( D' - D = div(\phi) \).

a). Prove that if \( X \) is projective, then \( \phi \) is unique up to scalar multiple.

b). Compute \(|H|\) where \( H \) is a hyperplane on \( \mathbb{P}^n \).

c). Compute \(|2L|\) where \( L \) is a line in \( \mathbb{P}^2 \).

d). In the notation of problem 6, show there is a natural bijection \( \mathbb{P}(\mathcal{O}_X(D)(X)) \to |D| \) when \( X \) is projective.

\(^4\)By definition, this sheaf is thus invertible.