1. **Cubic Surfaces.** Let $X \subset \mathbb{P}^3$ be the cubic surface defined by $V(x^2y + y^2z + z^2w + w^2x)$.

a). Show that $V(x,z)$ and $V(y,w)$ are two skew lines on $X$.

b). If $L_1$ and $L_2$ are coplanar lines lying on a cubic surface, explain why there is a third line on that surface in the same plane.

c). Is $X$ is smooth?\(^1\)

d). Find (a formula for an) explicit rational map (meaning, regular representative for a rational map) $X \dashrightarrow \mathbb{P}^2$.

2. **Rational Cubics of Even dimension.**

a). Using an argument similar to how we proved cubic surfaces are rational in class, prove that a cubic hypersurface of dimension $2n$ containing two skew $n$-planes is rational.

b). Find an explicit smooth cubic hypersurface in $\mathbb{P}^{2n+1}$ containing pair of skew $n$-planes.

c). Inside the space of all cubic hyper surfaces in $\mathbb{P}^{2n+1}$, show that the subset of cubics containing a fixed pair of skew $n$-planes is a proper closed set isomorphic to a projective space of a certain dimension. What dimension? Does this set intersect the open locus of smooth cubics?

d). Does the generic cubic hypersurface of even dimension (greater than two) contain two skew hyperplanes?

3. **Blowup along Principle ideals.** Recall that the blowup of an affine variety $X$ along an ideal $I$ generated by regular functions $f_0, \ldots, f_t$ is the graph of the rational map $X \dashrightarrow \mathbb{P}^t$ sending $x \mapsto [f_0(x) : \ldots : f_t]$.

Show that the blowup of any irreducible affine variety along a principle ideal is an isomorphism.

4. **Bad Blowing up.** Show that the blowup $X$ of $\mathbb{A}^2$ along the ideal $(x^2, y^3)$ is not smooth. Find defining equations for the singular locus of $X$ as a subvariety of $\mathbb{A}^2 \times \mathbb{P}^1$. (Thus bad blowing up can make the singularities of a variety get worse !)

5. **Blowing up a line.** a). Let $X$ be the blow-up of $\mathbb{A}^3$ along any line $L$. Let $\pi : X \dashrightarrow \mathbb{A}^3$ denote the blowing up morphism along $L$. Find defining equations for $X$ as a subvariety of $\mathbb{A}^3 \times \mathbb{P}^1$, and prove that $X$ is smooth.\(^2\)

b). The *exceptional set* of a birational morphism $\phi : X \rightarrow Y$ is the closed subset of $X$ at which $\phi$ is not an isomorphism. Describe the exceptional set of the blowup $\pi$ (from (a)) both geometrically and algebraically (by giving its defining equations.) What is its dimension?

c). Fix a point $p$ on $L$. Consider a line $\ell$ in $\mathbb{A}^3$ through $p$ other than $L$. Describe, both algebraically and geometrically, the point (or points) in $X$ in the fiber over $p$ which lie on the proper transform of the line $\ell$. (The proper transform of $\ell$ is the closure of $\pi^{-1}(\ell \setminus L \cap \ell)$ in $X$.)

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\(^1\)If you can code, or even use computer algebra packages, feel free to use technology for things like this—with careful citation how you did it!

\(^2\)Please do not torture yourself with poor choice of coordinates.
6. Nodal Curve Let $C = \mathbb{V}(y^2 - x^2 - x^3) \subset \mathbb{A}^2$. Let $\pi : \mathbb{A}^2 \to \mathbb{A}^2$ be the blowup of the origin in $\mathbb{A}^2$.

a). Show that $\pi^{-1}(C)$ is a reducible curve with two components, one isomorphic to $\mathbb{P}^1$.

b). Find the intersection points of the two components, expressing your answer in terms of coordinates on an appropriate affine patch of $\mathbb{A}^2$. Explain the geometric meaning of these intersection points.

c). The component of $\pi^{-1}(C)$ other than $\mathbb{P}^1$ is called the proper transform of $C$. Show that the proper transform of $C$ is equal to the (Zariski) closure of $\pi^{-1}(C - \{(0,0)\})$.

d). Show that the proper transform $\tilde{C}$ of $C$ is a non-singular variety, which is birationally equivalent to $C$. Describe the map $\pi : \tilde{C} \to C$. How does this compare to $\pi^{-1}(C) \to C$?

e). Illustrate all this with a picture, choosing a suitable affine patch of $\mathbb{A}^2 \times \mathbb{P}^1$.

7. Transverse intersection. a). Let $\{W_i\}_{i=1}^t$ be a finite set of subspaces of a finite dimensional vector space $V$. Show that $\operatorname{codim}(\bigcap_{i=1}^t W_i) \leq \sum_{i=1}^t \operatorname{codim}W_i$. We say that the $W_i$ intersect transversely if equality holds. Draw some pictures and ponder the meaning! For example: if $t > n$, what happens? if the $W_i$ are all hyperplanes, what does this mean? if $t = 2$, what typically happens? what else can you say?

b). Show that if $W \subset V$ is an inclusion of varieties, then there is an injective linear map $T_pW \subset T_pV$ for any $p \in W$.

c). Let $W_i$ be a finite collection of closed subvarieties of a smooth variety $V$, all containing a point $P$. Show that $\operatorname{codim}(\bigcap_{i=1}^t T_pW_i) \leq \sum_{i=1}^t \operatorname{codim}W_i$. We say that the $W_i$ intersect transversely if equality holds. Show that subvarieties of a smooth variety $V$ intersect transversely at $P$ if and only if each is smooth at $P$ and their tangent spaces at $P$ intersect transversely as subspaces of $T_pV$.

d). Explain what it means that two curves in $\mathbb{P}^2$ intersect transversely at a point $p$. Give examples of curves of any degree which intersect transversely at $p$. Give examples of smooth curves with no common components which do not intersect transversely at $p$.

8. Affine Schemes. Let $R$ be any commutative ring. Let $\operatorname{Spec}(R)$ denote the set of all prime ideals of $R$, and $\operatorname{mSpec}(R)$ denote the set of maximal ideals of $R$. (These are called the prime spectrum and the maximal spectrum of $R$, respectively).

a). Show that $\operatorname{Spec} \ R$ is a topological space with closed sets $\mathbb{V}(I) = \{P \mid P \supset I\}$, where $I$ is any ideal of $R$, and that the sets $U_f = \{P \mid f \notin P\}$ form a basis.

b). Show the inclusion of sets $\operatorname{mSpec}(R) \subset \operatorname{Spec} \ R$ induces an homeomorphism between $\operatorname{mSpec}(R)$ (with induced subspace topology) and the subspace of closed points in $\operatorname{Spec} \ R$.

c). Let $R$ be the coordinate ring of an affine variety $X$ (over an algebraically closed field of course). Show that $X$ is homeomorphic to $\operatorname{mSpec} \ R$. What do the non-closed point of $\operatorname{Spec} \ R$ add to the picture?

d). For any ring homomorphism $f : R \to S$, show that there is an continuous map of topological spaces $\operatorname{Spec} \ S \to \operatorname{Spec} \ R$, sending each point $P$ in $\operatorname{Spec} \ S$ to $f^{-1}(P)$ in $\operatorname{Spec} \ R$. If $R$ and $S$ are reduced finitely generated algebras over an algebraically closed field, explain (with proof!) the significance of this map in light of question c.

e). For a Noetherian ring $R$, let $N$ be the nil radical of $R$. Show that the natural surjection $R \to R/N$ induces an homeomorphism of topological spaces. For $R$ a finitely generated algebra over an algebraically closed field, explain how there is a naturally underlying variety homeomorphic to $\operatorname{maxSpec} \ R$. [However, we should think of $\operatorname{Spec} \ R$ and even $\operatorname{maxSpec} \ R$ something more than this underlying variety, since its “functions” include all elements of $R$, not just elements of $R/N$. Ponder this.]

f). Describe $\operatorname{Spec} k[x, y]/(x^2, y)$ and $\operatorname{Spec} k[x, y]/(x, y^2)$ both as topological spaces, and in terms of having something “extra.” Can you interpret the “extra” both algebraically and geometrically? In what sense are these gadgets the same or different? How do they compare to $\operatorname{Spec} k[x, y]/(x, y)$?