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On Langlands' automorphic Galois group & its approximations:

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- $F$ number field, $G/F$ q.s, $\pi$ irred. admiss. rep. of $G(\mathbb{A})$

$\rightsquigarrow$ $c(\pi) = \{c_v(\pi) = c(\pi_v) : v \notin S\}$, $c_v(\pi)$ Frob-Hecke conj class in ${}^LG_v \subset {}^LG$. (F-H class)

- Write $c = \{c_v : v \notin S\} \sim c' = \{c'_v : v \notin S'\}$ if $c_v = c'_v$ for a.a. $v$,

+ set $\mathcal{C}_{aut}(G) = \{c(\pi) : \pi \in \Pi_{aut}(G)$ – automorphic$\}$ – eq. classes –

and $\mathcal{C}_{sim}(G) = \{c(\pi) \in \mathcal{C}_{aut}(G) : \pi$ cuspidal$\} \subset \mathcal{C}_{aut}(G)$,

in case $G = GL(N)$ – concrete data.

GENERAL RESULTS: Set $\Pi(G) = \{\pi \subset L^2(G(F)\backslash G(\mathbb{A})) \subset \Pi_{aut}(G)$
(occurs in spec. decomp)

1. (Harris-Taylor, Henniart, Scholze; Jacquet-Shalika; Moeglin-Waldspurger)
A classification of $\Pi(G)$, $G = GL(N)$ in terms of sets $\mathcal{C}_{sim}(N) = \mathcal{C}_{sim}(GL(N))$

2. (A.——) Classification of $\Pi(G)$, $G$ orthog. or symplectic in terms of sets $\mathcal{C}_{sim}(N)$

3. (Mok) Classification of $\Pi(G)$, $G$ unitary, in terms of sets $\mathcal{C}_{sim}(N)$ (with $F$ replaced by $E$, $\deg(E/F) = 2$)

REMARKS: (i) Takes care of q.s. groups in each of the four infinite families $\underline{\underline{A}}_n, \underline{\underline{B}}_n, \underline{\underline{C}}_n, \underline{\underline{D}}_n$

(ii) Class$^{\underline{ns}}$ are in terms of local packets $\Pi_{\psi_v}$ for $G_v$, global packets $\Pi_\psi$ for $G$ + mult. formula for any $\pi \in \Pi_\psi \in L^2_{disc}(G(F)\backslash G(\mathbb{A}))$. In particular, includes local Langlands class$^{\underline{n}}$. ~~is fund. in underlying~~

(iii) The sets $\mathcal{C}_{sim}(N)$ represent the fund. underlying data.

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QUESTIONS: (i) There are sets more fundamental than $\Pi_{sim}(N)$. What are they?

(ii) Global class$^{\underline{n}}$ is stated in terms of a crude subst. for Langlands hypothetical aut. Galois gp $L_F$. Can we refine it?

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Recall: Principal of Functoriality (Langlands)

$G, G'/F$, $\rho: {}^LG' \longrightarrow {}^LG$ an L-homo$^{\underline{sm}}$

Then if $c' = \{c'_v\}$ lies in $\mathcal{C}_{aut}(G')$, the family $c = \rho(c') = \{\rho(c'_v)\}$ lies in $\mathcal{C}_{aut}(G)$

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e.g. Suppose $G = GL(N)$ + ${}^LG' \subset {}^LG$ is proper sub$^{\underline{gp}}$ + $c \in \mathcal{C}_{sim}(N)$ is image of $c' \in \mathcal{C}_{aut}(G')$. Then $c'$ is clearly more basic than $c$. Describe most fundamental of these?

<u>Recall</u>: If $G$ is arbitrary (q.s.), $c \in \mathcal{C}_{aut}(G)$, and
$r : {}^LG \longrightarrow GL(N, \mathbb{C})$, we set

$$L^S(s, c, r) = \prod_{v \notin S} \det(1 - r(c_v)|\varpi_v|^{-s})^{-1}.$$

Then functoriality $\Rightarrow$ $L^S(s, c, r)$ has an. cont. & $f^n$ eq$^n$.

⌈Remark on Artin L $f^{ns}$ for $r : W_F \longrightarrow GL(N, \mathbb{C})$ where analogue of functoriality is trivial.⌋

Assume functoriality for present & assume q.s. $G/F$ is simple & s.c. Define $\mathcal{C}_{prim}(G)$ to be set of $c \in \mathcal{C}_{aut}(G) \ni$

$$\mathrm{ord}_{s=1}(L^S(s, c, r)) = -[r : 1_{{}^LG}] \sim \text{mult. of triv. rep. in } r.$$

⌈$\mathcal{C}_{prim}(G)$ should also be set of $c \in \mathcal{C}_{aut}(G)$ that are <u>primitive</u>, in sense they are not proper func. images; ~~[illegible]~~ not known if this is equiv. condition – even with functoriality; should follow from <u>proof</u> of functoriality along lines of beyond endoscopy – i.e. using trace formula.⌋

$\mathcal{C}_{prim}(G)$ are really the "fund." objects! set of but ~~that should be the hyp. aut. Galois gp.~~

⌈For $G = GL(N)$, get

$$\begin{array}{ccccccc} \mathcal{C}_{prim}(N) & \subset & \mathcal{C}_{sim}(N) & \subset & \mathcal{C}(N) & \subset & \mathcal{C}_{aut}(N) \\ & & \wr\wr & & \wr\wr & & \wr\wr \\ & & \Pi_{cusp}(N) & & \Pi(N) & & \Pi_{aut}(N) \end{array}$$ ⌋

<u>Def</u>: Let $\mathcal{C}_F$ be set of iso$^{\text{bm}}$ classes of pairs

$$\{(G,c) : G/F \text{ q.s., simple, sc}\ ;\ c \in \mathcal{C}_{\text{prim}}(G)\}$$

<u>Goal</u>: Build a loc cpt gp $L_F$ out of $\mathcal{C}_F$ that should be the hyp. aut. Galois group.

Suppose $(G,c) \in \mathcal{C}_F$. Let $K_c$ be cpt real form of $\hat{G}_{sc}$ – a <u>cpt</u>, <u>sc</u> gp. The contrib. of $(G,c)$ to $L_F$ will be an ext$^{n}$

$$(*) \qquad 1 \longrightarrow K_c \longrightarrow L_c \longrightarrow W_F \longrightarrow 1.$$

To define $(*)$, let

$$1 \longrightarrow Z \xrightarrow{\ \varepsilon\ } \tilde{G} \longrightarrow G_{ad} \longrightarrow 1$$

be a z-ext$^{n}$ of $G_{ad}$ (so $G = \tilde{G}_{der} \subset \tilde{G}$ & $Z$ is ind$^{d}$ torus), & set $\widehat{K}_c = \text{Norm}(K_c, \hat{\tilde{G}})$

Typical example:
- $G = SL(N)$, $G_{ad} = PGL(N)$, $\tilde{G} = GL(N)$, $Z = \mathbb{G}_m$,
- $K_c = SU(N) \subset SL(N,\mathbb{C}) = \hat{G}_{sc}$
- $\widetilde{K}_c = U(N) \subset GL(N,\mathbb{C}) = \hat{\tilde{G}}$.

Hypothesis (not deep): $\exists\ \tilde{c} \in \mathcal{C}_{\text{aut}}(\tilde{G})$ whose image under dual mapping ${}^L\tilde{G} \longrightarrow {}^LG = ({}^L\tilde{G})_{ad}$ is the given $c \in \mathcal{C}_{\text{prim}}(G)$

i.e. $\tilde{c} = c(\tilde{\pi})$ for some $\tilde{\pi} = \tilde{\pi}_c$ in $\Pi_{\text{aut}}(\tilde{G})$.

Consider the dual ext$^{n}$

$$1 \longrightarrow \hat{G}_{sc} \longrightarrow \hat{\tilde{G}} \xrightarrow{\ \hat{\varepsilon}\ } \hat{Z} \longrightarrow 1.$$

(5)

- This gives $\hat{\varepsilon}(\hat{c}) \in \mathcal{C}_{aut}(Z)$, with $\hat{\varepsilon}(c) = c(\hat{\chi})$, where $\hat{\chi} = \hat{\chi}_c \in \Pi_{aut}(Z)$ is central char. of $\pi_c$. ~~[illegible]~~
- Then let $z_c : W_F \to \hat{Z}$ be cocycle dual to $\chi_c$ (Langlands global corresp for torus $Z$) + set

$$L_c = \{ g \rtimes w \in \hat{K}_c \rtimes W_F \subset {}^L\hat{G} \; : \; \hat{\varepsilon}(g) = z_c(w) \}.$$

This is the ext$^{=}$ (*).

- Given (*) for all $(G,c) \in \mathcal{C}_F$, define fibre product

$$L_F = \prod_{(G,c) \in \mathcal{C}_F} (L_c \longrightarrow W_F)$$

– an ext$^{=}$ of $W_F$ by cpt, sc gp $K_F = \prod_{(G,c)} K_c$.

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If we assume local Langlands ~~for local constit~~ $\pi_v$ (for each $c \longrightarrow \hat{c} = c(\pi) \longrightarrow \pi$ as above), we easily get local embeddings

$$\begin{tikzcd}
L_{F_v} \arrow[r] \arrow[d, hook] & W_{F_v} \arrow[r] \arrow[d, hook] & Gal(\bar{F}_v/F_v) \arrow[d, hook] \\
L_F \arrow[r, two heads] & W_F \arrow[r, two heads] & Gal(\bar{F}/F)
\end{tikzcd},$$

where

$$L_{F_v} = \begin{cases} W_{F_v}, & v \text{ archimed.} \\ W_{F_v} \times SU(2), & v \text{ p-adic} \end{cases}$$

is loc. Langlands gp.

<u>Conjecture</u>: $L_F$ is the aut. Galois group $(\cong \mathcal{C}_{sim}(N))$: i.e. $\forall N$, there is a canonical bijection $\Phi_{sim}(N) \xrightarrow{\sim} \Pi_{cusp}(N)$, where $\Phi_{sim}(N)$ is the set of equiv. classes of irred rep$^s$

$$\phi : L_F \longrightarrow GL(N, \mathbb{C}).$$

<u>Corollary</u> (Of General Results (1), (2) + (3)) Assume the conjecture. Then for $G$ as in (1), (2) or (3), the global packets $\Pi_\psi$ for $L^2_{disc}(G(F) \backslash G(\mathbb{A}))$ are canonically parametrized by $\hat{G}$-orbits of L-homo$^{sm}$

$$\psi : L_F \times SU(2) \longrightarrow {}^L G$$

with $|\,\mathrm{Cent}(\mathrm{Im}(\psi), \hat{G}) / Z(\hat{G})\,| < \infty$.

<u>Approximation of $L_F$</u>: Define $\Phi_F = \coprod_{N \geq 1} \Phi_{sim}(N)$, the set of irred. rep$^s$ of $L_F$

<u>Problem</u> (i) (For $GL(N)$) Replace hypothetical pair $(L_F, \Phi_F)$ by an explicit, <u>unconditional</u> pair $(L_F^*, \Phi_F^*)$, for a loc. cp$^t$ extension

$$1 \longrightarrow K_F^* \longrightarrow L_F^* \longrightarrow W_F \longrightarrow 1,$$

+ a distinguished set $\Phi_F^*$ of finite dim. rep$^s$ of $L_F^*$ that serves as subst. for $L_F$ for aut. rep$^s$ of $GL(N)$.

i.e. so that there is a (hypothetical) L-embedding

$$\iota : L_F \longrightarrow L_F^*,$$

defined up to $K_F^*$-conj. ∋ the rest$^n$ mapping

$$\phi^* \longrightarrow \phi = \phi^* \circ \iota, \qquad \phi^* \in \Phi_F^*,$$

is a bijection from $\Phi_F^*$ to $\Phi_F$.

(ii) (For $G$ as in (1), (2) + (3)). Replace $(L_F^*, \Phi_F^*)$ by a refinement $(\widetilde{L}_F^*, \widehat{\Phi}_F^*)$, with L-embedding

$$L_F \xrightarrow{\widetilde{\iota}} L_F^* \xrightarrow{\widetilde{\iota}^*} L_F^*, \qquad \widetilde{\iota}^* \circ \widetilde{\iota}^* = \iota,$$

+ $\widehat{\Phi}_F^*$ the set of self-dual rep$^s$ in $\Phi_F^*$, that plays role of $L_F$ for aut. rep$^s$ of any $G$ as in (1), (2), (3)

Reasons why

(i) Give clean statement of results in General Theory (2).

(ii) Instructive: concrete analogue of const$^n$ of $L_F$ above.

(iii) Instructive: raises new (?) quest$^{ns}$ for cyclic base change for GL(N)

Rough Idea: Replace $\mathcal{C}_F = \{(G, c)\}$ by

$$\mathcal{C}_F^* = \{(N, c) : N \geqslant 1,\ c \in \mathcal{C}_{sim}^*(N)\} \doteq \mathcal{C}_{sim}(N) / \mathcal{C}_{sim}(1)$$
$$= \{c(\pi) : \pi \in \Pi_{cusp}(N)\} / \mathcal{C}_{sim}(1),$$

with special att$^n$ to its self-dual subset

$$\widehat{\mathcal{C}}_F^* = \{(N, c) : N \geqslant 1,\ c \in \widehat{\mathcal{C}}_{sim}^*)\} \doteq \widehat{\mathcal{C}}_{sim}(N) / \widehat{\mathcal{C}}_{sim}(1)$$
$$= \{c = c(\pi) \in \mathcal{C}_{sim}(N) : c = c^\vee\} / \widehat{\mathcal{C}}_{sim}(1).$$