A Plancherel Formula for Almost Symmetric Spaces

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Suppose G reductive group , H reductive subgroup.

Disintegrate the representation of G on $L^2(G/H)$.

Why ?

- Central theme in harmonic analysis: Harish Chandra for H=e.

  Recommended Reading

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THE PLANCHEREL FORMULA, THE PLANCHEREL THEOREM, AND THE FOURIER TRANSFORM OF ORBITAL INTEGRALS

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Abstract. We discuss various forms of the Plancherel Formula and the Plancherel Theorem on reductive groups over local fields.

Dedicated to Gregg Zuckerman on his 60th birthday

1. Introduction

The classical Plancherel Theorem proved in 1910 by Michel Plancherel can be stated as follows:

Theorem 1.1. *Let $f \in L^2(\mathbb{R})$ and define $\phi_n : \mathbb{R} \to \mathbb{C}$ for $n \in \mathbb{N}$ by*

$$\phi_n(y) = \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} f(x) e^{iyx} dx.$$

The sequence ϕ_n is Cauchy in $L^2(\mathbb{R})$ and we write $\phi = \lim_{n\to\infty} \phi_n$ (in L^2). Define $\psi_n : \mathbb{R} \to \mathbb{C}$ for $n \in \mathbb{N}$ by

$$\psi_n(x) = \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} \phi(y) e^{-iyx} dy.$$

The sequence ψ_n is Cauchy in $L^2(\mathbb{R})$ and we write $\psi = \lim_{n\to\infty} \psi_n$ (in L^2). Then,

$$\psi = f \text{ almost everywhere, and } \int_{\mathbb{R}} |f(x)|^2 \, dx = \int_{\mathbb{R}} |\phi(y)|^2 \, dy.$$

This theorem is true in various forms for any locally compact abelian group. It is often proved by starting with $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, but it is really a theorem about square integrable functions.

There is also a "smooth" version of Fourier analysis on $\mathbb{R}$, motivated by the work of Laurent Schwartz, that leads to the Plancherel Theorem.

Definition 1.2 (The Schwartz Space)**.** The *Schwartz space*, $\mathcal{S}(\mathbb{R})$, is the collection of complex-valued functions f on $\mathbb{R}$ satisfying:

(1) $f \in C^\infty(\mathbb{R})$.
(2) f and all its derivatives vanish at infinity faster than any polynomial. That is, $\lim_{|x|\to\infty} |x|^k f^{(m)}(x) = 0$ for all $k, m \in \mathbb{N}$.

Fact 1.3. *The Schwartz space has the following properties:*

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For nontrivial H and the left regular representation on $L^2(G/H)$

- Helgason for H maximal compact subgroup,

- H= fix point set of an involution work of van den Ban , Schlichtkrull, Delorme, Oshima ,

- In other special cases by many others.

Other interesting perspectives:

- Burger, Li, Sarnak :
  $\pi \in L^2(G/H)_{dis}$ implies $\pi \in L^2(G/\Gamma)_{dis}$ for some Γ

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- Conversely understanding $L^2(G/\Gamma)_{dis}$ gives us information about $L^2(G/H)$. (For example G classical and quasi split)

- Benoist + Kobayashi recently considered(solved) the problem: Find a condition so that all representations in $L^2(G/H)$ tempered, but they obtained no information about the multiplicities. Interesting to look at cases where $L^2(G/H)$ is not tempered and find the multiplicities of the tempered representations.

The following is joint work in progress with Bent Ørsted.

We consider a noncompact subgroup $H = H_{ss}Z_H$ where H is a subgroup of finite index in the fix points of an involution of G and $Z_{ss} = \mathbb{R}$ is a subgroup of finite index of the center of H. We call G/H_{ss} an almost symmetric space.

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Theorem 1. *Suppose that G/H is an almost symmetric space. As a left regular representation of G*

$$L^2(G/H_{ss}) = L^2(G/H) \cdot L^2(Z_H).$$

Corollary 2. *All irreducible representations in the discrete spectrum of* $L^2(G/H_{ss})$ *have infinite multiplicity.*

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Corollary 3. $L^2(G/H_{ss})$ *is tempered iff* $L^2(G/H)$ *is tempered.*

An Example:
$G = SL(2,\mathbb{R})$, H diagonal matrices Then $X = G/H$ is a hyperboloid and

$$L^2(G/H) = \oplus_\nu \textit{ even } D_\nu + 2\int_0^\infty \pi_{it}$$

where D_ν are the discrete series representations with parameter ν and π_{it} are the tempered principal series representations with parameter it.

Here $H_{ss} = \mathbb{Z}_2$, then $L^2(G/H_{ss}) = L^2(PSL(2,\mathbb{R}))$ and so the left regular representation contains the even discrete series representations with ∞ multiplicity.

If H is connect then $L^2(G/H)$ contains all discrete series representations and so does the left regular representation of G on $L^2(G)$.

Example 2: $G = SL(2n, \mathbb{R})$, We take H as the connected component of $S(GL(p, \mathbb{R})GL(q, \mathbb{R})$. Then $H_{ss} = SL(p, \mathbb{R})SL(q, \mathbb{R})$ where $p + q = 2n$.

- If p=q =n then $L^2(SL(2n, \mathbb{R}/H)$ is tempered .

- If $p - q \geq 2$ then $L^2(Sl(2n, \mathbb{R})/SL(p, \mathbb{R}) \times SL(q, \mathbb{R}))$ is not tempered.

- Using induction by stages we get Let $m < n$ we deduce $L^2(Sl(2n, \mathbb{R})/SL(m, \mathbb{R})SL(m, \mathbb{R}))$ is tempered.

Example 3: $G = SL(2n,\mathbb{C})$, H_{ss} has a covering $T^1 SL(p,\mathbb{C}) \times SL(q,\mathbb{C})$, p+q = 2n with a one dimensional torus T^1.

Then

$$L^2(SL(n,\mathbb{C})/SL(p,\mathbb{C}) \times SL(q,\mathbb{C})) = \oplus_{\delta \in \widehat{T}} L^2(SL(n,\mathbb{C})/H_{ss},\delta)$$

where $L^2(SL(n,\mathbb{C})/H_{ss},\delta)$ are the L^2-section of the line bundle defined by the character δ of H_{ss}.

Result:

- If p=q =n then $L^2(SL(2n,\mathbb{C}/H_{ss})$ is tempered

- If $p - q \geq 2$ then $L^2(Sl(2n,\mathbb{C})/SL(n,\mathbb{C}) \times SL(n,\mathbb{C}))$ is not tempered.

Example 4: Cayley type spaces considered in Olafson- Ørsted.

1. $G = Sp(n,\mathbb{R})$, $H = GL(n,\mathbb{R})$ and $H_{ss} = SL_{+/-}(n,\mathbb{R})$, $n > 1$

2. $G = SO(2,n)$, $H = SO(1,1)SO(1,n-1)$ and $H_{ss} = SO(1,n-1)$, $n > 2$

3. $G = SU(n,n)$, $H = SL(n,\mathbb{C})\mathbb{R}^+$ and $H_{ss} = SL(n,\mathbb{C})$

4. $G = O^*(4n)$, $H = \mathbb{R}^+ SU^*(2n)$ and $H_{ss} = SU^*(2n)$

5. $G = E_7(-25)$, $H = E_6(-26)\mathbb{R}^+$ and $H_{ss} = E_6(-26)$

Results: If n is large enough then

- $L^2(Sp(n,\mathbb{R})/SL(n,\mathbb{R})$ and $L^2(E_7(-25)/E_6(-26))$ are tempered
- $L^2(SO(2,n)/SO(1,n-1)$, $L^2(SU(n,n)/SL(n,\mathbb{C}))$ and $L^2(O^*(4n)/SU^*$ are not tempered.

All representations have infinite multiplicity in Plancherel formula.

Proof of the theorem:

1. Step:

$H \subset G$ a subgroup with finite number of connected components and $H = H_s Z_H$ with $Z_H = \mathbb{R}^+$ in the center of H.

We extend a character $\chi \in \widehat{Z_H}$ to a character of H and consider the induced representation Ind ${}_H^G \chi$ on $L^2(G/H)_{\chi^{-1}}$.

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Lemma 4. *As a representation of* G

$$L^2(G/H_{ss}) = \int_{\chi \in \widehat{Z_H}} L^2(G/H)_{\chi^{-1}} d\chi$$

Proof: Uses Fourier analysis on Z_H and is not difficult..

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Main Problem:

Let χ and $\tilde{\chi}$ be characters of Z_H considered as characters of H. Show that

$$Ind_H^G \chi = Ind_H^G \tilde{\chi}.$$

Now have to assume that

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Observation

We may consider H as a subgroup of finite index in the Levi subgroup of a maximal parabolic subgroup $P = LN$ with abelian unipotent radical N.

Let $\chi \in \widehat{Z_H}$. We consider again χ as a character of H and consider the unitary induced representation. $\mathrm{Ind}_H^P \chi$.

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Sketch of the proof:
We denote the induced representations act on functions $F \in L^2(N)$ by

$$\rho_\chi(n_0)F(n) = F(n \cdot n_0)$$

$$\rho_\chi(h_0)F(n) = \chi(h_0)F(h_0^{-1} n h_0)$$

Using the Fourier transform on $L^2(N)$ we realize the representation $Ind_H^P \chi$ on $L^2(\widehat{N})$ by

$$\widehat{\rho}_\chi(n_0) \text{ is a multiplication operator}$$

$$\widehat{\rho}_\chi(h_0)\widehat{F}(\xi) = \chi(h_0)J(h_0^t\xi)^{1/2}\widehat{F}(h_0^t\xi)$$

Now $\widehat{N}$ is the closure of a finite number of open orbits $\mathcal{O}_i$ of L on $\widehat{N}$ and so the representation $\widehat{\rho}_\chi$ is a direct sum of representations on

$$\oplus_i L^2(\mathcal{O}_i)$$

On each summand we have an intertwining operator (dependent on the orbit)

$$I_i(\chi, \tilde{\chi}) : \widehat{\rho}_\chi \to \widehat{\rho}_{\tilde{\chi}}.$$

Example: Consider the group $P = HN$ with

$$H = \{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad | \ a > 0\}$$

and

$$N = \{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad | \ b \in \mathbb{R}\}$$

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There are 3 orbits of H on

$$\hat{N} = \{\xi_t \mid \xi_t(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}) = e^{it \cdot b}\}.$$

namely $\mathcal{O}^+ = \{\xi_t \mid t > 0\}$, $\mathcal{O}^- = \{\xi_t \mid t < 0\}$ and $\mathcal{O}^1 = \{\xi_0\}$.

The unitary representation ρ_1 of P induced from the trivial representation of H acts on $L^2(N)$ by

$$\rho_1(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})F(x) = a^{1/2}F(ax)$$

and

$$\rho_1(\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix})F(x) = F(x+b).$$

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The representation ρ_1 is a direct sum of 2 unitary representations on square integrable functions whose Fourier transform has support in $\xi \in \mathcal{O}^+$, respectively in $\xi \in \mathcal{O}^-$.

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The representation ρ_1 is a direct sum of 2 unitary representations on square integrable functions whose Fourier transform has support in $\xi \in \mathcal{O}^+$, respectively in $\xi \in \mathcal{O}^-$.

Consider $\chi_s : a \to a^{is}$ as a character of H and consider the representation $\widehat{\rho_s}$ induced from χ_s.

After applying the Fourier transform the representation $\widehat{\rho_s}$ has the form

$$\widehat{\rho_s}(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})\widehat{F}(\xi) = a^{-1/2}a^{is}\widehat{F}(a^{-1}\xi)$$

and

$$\widehat{\rho_s}(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix})\widehat{F}(\xi) = e^{ib\xi}\widehat{F}(\xi)$$

The equivalence of the representations $r\widehat{h}o_s$ and $\widehat{\rho}_1$ follows from the intertwining operator

$$\mathcal{I}_s : \widehat{\rho}_0 \to \widehat{\rho}_s \qquad \text{defined by} \qquad \mathcal{I}_s\widehat{F}(\xi) = \xi^{is}\widehat{F}(\xi).$$

Induction to G

Proposition 6. *Let* χ *and* $\tilde{\chi}$ *be characters of* Z_H *considered as characters of* H*. Under we have*

$$Ind_H^G \chi = Ind_H^G \tilde{\chi}.$$

Proof. By induction by stages

$$Ind_H^G \chi = ind_P^G Ind_H^P \chi = ind_P^G Ind_H^P \tilde{\chi} = Ind_H^G \tilde{\chi}$$

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I conjecture that this proposition is true if $L = HA$ is the Levi subgroup of a "very nice " parabolic subgroup in N. Wallach's terminology.

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Theorem 7. *Suppose that G/H is an almost symmetric space. As a left regular representation of G*

$$L^2(G/H_{ss}) = (Ind_H^G 1) \cdot L^2(Z_H) = L^2(G/H) \cdot L^2(Z_H).$$

Proof. This follows from previous proposition.

Happy birthday Becky