Square Integrable Representations

Midwest Representation Theory Conference

University of Chicago

September 5–7, 2014

Joseph A. Wolf

University of California at Berkeley

Heisenberg Group

- $\langle w, w' \rangle$ is the standard hermitian inner product on $\mathbb{C}^n$
- $\mathfrak{h}_n = i\mathbb{R} + \mathbb{C}^n$ Heisenberg alg, $[z + w, z' + w'] = \operatorname{Im} \langle w, w' \rangle$
- $H_n = i\mathbb{R} + \mathbb{C}^n$ Heisenberg group: Lie algebra $\mathfrak{h}_n$
- H_n has center $Z = i\mathbb{R}$, $\mathfrak{h}_n$ has center $\mathfrak{z} = i\mathbb{R}$
- Each $\mathbb{R}$–linear functional $\xi : \mathbb{C}^n \to \mathbb{R}$ defines a unitary character $\chi_\xi : z + w \mapsto \exp(2\pi i \xi(w))$ on H_n
- $0 \neq \lambda \in \mathfrak{z}^*$ defines an infinite dimensional irreducible unitary representation π_λ of H_n with $\pi_\lambda|_Z = \exp(2\pi i \lambda)$
- Uniqueness of the Heisenberg commutation relations says that every irreducible unitary representation of H_n is equivalent to a χ_ξ if it annihilates Z, to a π_λ if it does not
- Fourier inversion has form $f(x) = c_n \int_{\mathfrak{z}^*} \Theta_{\pi_\lambda}(r_x f) |\lambda|^n d\lambda$

Kirillov Theory

- Kirillov used representation theory of H_n to give a general theory of unitary reps of csc nilpotent Lie groups
- N is a csc Lie group, $\mathfrak{n}$ its Lie algebra, $\mathfrak{n}^*$ dual space of $\mathfrak{n}$
- If $f \in \mathfrak{n}^*$: coadjoint orbit $\mathrm{Ad}^*(N)f$ has invariant symplectic form ω_f from $b_f : \mathfrak{n} \times \mathfrak{n} \to \mathbb{R}$, $b_f(x, y) = f([x, y])$.
- *polarization*: subalgebra $\mathfrak{p} \in \mathfrak{n}$ s.t. $ker\, b_f \subset \mathfrak{p} \subset \mathfrak{n}$ and $\mathfrak{p}/ker\, b_f$ maximal null (Lagrangian) subspace of $\mathfrak{n}/ker\, b_f$
- $\chi_f : \exp(x) \mapsto e^{2\pi i f(x)}$ unitary character on $P = \exp(\mathfrak{p})$
- That defines an (irreducible) unitary rep $\pi_f = \mathrm{Ind}\,_P^N(\chi_f)$
- π_f depends (to unitary equiv) only on the orbit $\mathrm{Ad}^*(N)f$
- Every irreducible unitary rep of N is equiv to some π_f
- Summary: bijection $\widehat{N} \leftrightarrow \mathfrak{n}^*/\mathrm{Ad}^*(N)$

Heisenberg Group Case

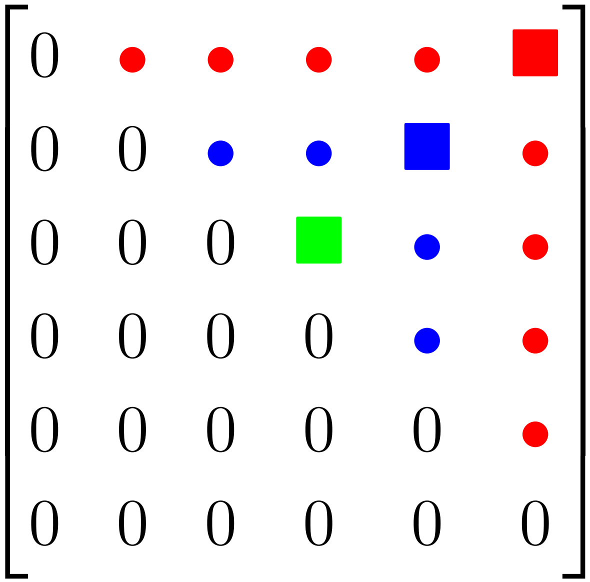
- $H_n = i\mathbb{R} + \mathbb{C}^n$ center $Z = i\mathbb{R}$, $\mathfrak{h}_n = i\mathbb{R} + \mathbb{C}^n$ center $\mathfrak{z} = i\mathbb{R}$
- unitary characters $\chi_\xi(z+w) = \exp(2\pi i \xi(w))$ for $\xi \in \mathfrak{h}_n^*$ with $\xi|_\mathfrak{z} = 0$ (i.e. $\xi(z+w) = \xi(w)$) and $\mathrm{Ad}^*(N)\xi = \{\xi\}$.
- infinite dimensional irreducible unitary representations $\pi_\lambda = \mathrm{Ind}\,_P^N(\exp(2\pi i \lambda|_\mathfrak{p}))$ with $0 \neq \lambda \in \mathfrak{z}^*$ extended to $\mathfrak{h}_n$ by $\lambda(\mathbb{C}^n) = 0$. Here $\mathrm{Ad}^*(N)\lambda = \{\nu \in \mathfrak{n}^* \mid \nu|_\mathfrak{z} = \lambda|_\mathfrak{z}\}$
- the coefficients $f_{u,v}(g) = \langle u, \pi_\lambda(g)v\rangle$ of π_λ satisfy $|f_{u,v}| \in L^2(N/Z)$.
- the Fourier transform is $\widehat{f}(\lambda) = \mathrm{trace}\ \int_N f(g)\pi_\lambda(g)dg$ for $f \in C_c^\infty(N)$ (or even for $f \in \mathcal{S}(N)$ Schwartz space)
- the Fourier inversion formula is $f(g) = c\int_{\mathfrak{z}^*} \widehat{f}(\lambda)|\lambda|^n d\lambda$ where c depends only on normalization of measures

Moore – W. Theory

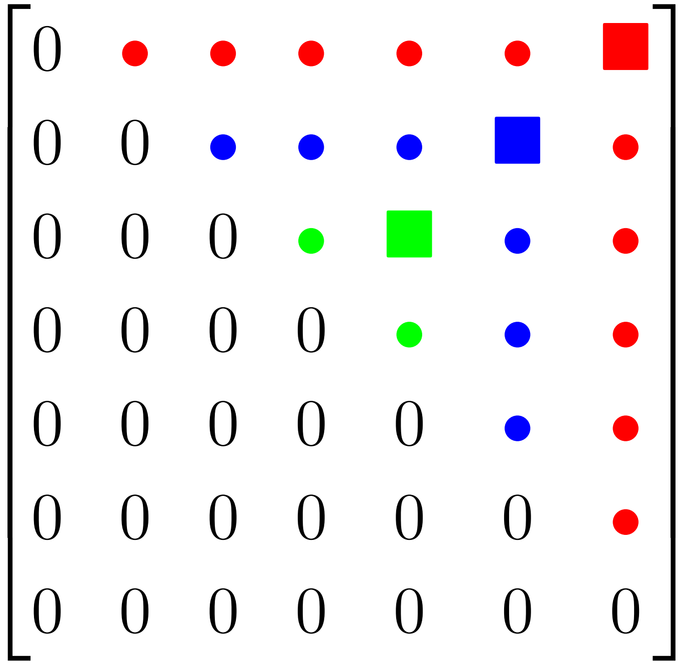
- Moore and W. simplified Kirillov theory for csc nilpotent Lie groups with square integrable (modulo center) representations, e.g. Heisenberg group and many others
- Let N be a csc nilpotent Lie group, $\mathfrak{n} = \mathfrak{z} + \mathfrak{v}$ vector space direct sum where $\mathfrak{z}$ is its center, $\mathfrak{n}^* = \mathfrak{z}^* + \mathfrak{v}^*$
- $P : \mathfrak{z}^* \to \mathbb{R}$ is the polynomial $P(\lambda) = \mathrm{Pf}(b_\lambda)$, where $\mathrm{Pf}(b_\lambda)$ is the Pfaffian of the antisymmetric form b_λ on $\mathfrak{n}/\mathfrak{z}$
- The following are equivalent for $\lambda \in \mathfrak{n}^*$:
  - 1. $\mathrm{Ad}^*(N)\lambda = \{\nu \in \mathfrak{n}^* \mid \nu|_\mathfrak{z} = \lambda|_\mathfrak{z}\}$
  - 2. $\pi_\lambda \in \widehat{N}$ has coefficients in $L^2(N/Z)$
  - 3. $P(\lambda) \neq 0$
- the Fourier inversion formula is $f(g) = c \int_{\mathfrak{z}^*} \widehat{f}(\lambda)|\mathrm{Pf}(\lambda)|d\lambda$ where c depends only on normalization of measures

Upper Triangular Matrices 1

- We foliate the upper triangular matrices:

- 

  or

- Red indicates a normal subgroup L_1 that is a Heisenberg group (the square is its center); blue is a subgroup L_2 that is a Heisenberg group (the square is its center); green is a subgroup L_3 that is a Heisenberg (or abelian) and the square is its center.

Upper Triangular Matrices 2

- $N = \begin{bmatrix} 0 & \bullet & \bullet & \bullet & \bullet & \blacksquare \\ 0 & 0 & \bullet & \bullet & \blacksquare & \bullet \\ 0 & 0 & 0 & \blacksquare & \bullet & \bullet \\ 0 & 0 & 0 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ or $N = \begin{bmatrix} 0 & \bullet & \bullet & \bullet & \bullet & \bullet & \blacksquare \\ 0 & 0 & \bullet & \bullet & \bullet & \blacksquare & \bullet \\ 0 & 0 & 0 & \bullet & \blacksquare & \bullet & \bullet \\ 0 & 0 & 0 & 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
- More generally this gives a decomposition

  $N = L_1 L_2 \dots L_{m-1} L_m$ where

  (a) each L_r has unitary reps with coef. in $L^2(L_r/Z_r)$,

  (b) each $N_r := L_1 L_2 \dots L_r$ is a normal subgp of N with

  $N_r = N_{r-1} \rtimes L_r$ semidirect product decomposition,

  (c) Let $\mathfrak{l}_r = \mathfrak{z}_r + \mathfrak{v}_r$ and $\mathfrak{n} = \mathfrak{s} + \mathfrak{v}$ vector space direct

  sums, $\mathfrak{s} = \bigoplus \mathfrak{z}_r$, and $\mathfrak{v} = \bigoplus \mathfrak{v}_r$.

  Then $[\mathfrak{l}_r, \mathfrak{z}_s] = 0$ and $[\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{v}$ for $r > s$.

Construction of Representations

- $N = L_1 L_2 \ldots L_{m-1} L_m$ where (a) each L_r has unitary reps with coef. in $L^2(L_r/Z_r)$, (b) each $N_r := L_1 L_2 \ldots L_r$ is a normal subgp of N with $N_r = N_{r-1} \rtimes L_r$ semidirect, (c) $\mathfrak{l}_r = \mathfrak{z}_r + \mathfrak{v}_r$, $\mathfrak{v} = \oplus \mathfrak{v}_r$, $[\mathfrak{l}_r, \mathfrak{z}_s] = 0$ and $[\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{v}$ for $r > s$.
- $\lambda_1 \in \mathfrak{z}_1^*$ with $P_{\mathfrak{l}_1}(\lambda_1) \neq 0$ gives $\pi_{\lambda_1} \in \widehat{L_1}$
- Then $\lambda_2 \in \mathfrak{z}_2^*$ with $P_{\mathfrak{l}_2}(\lambda_2) \neq 0$, and $\pi_{\lambda_2} \in \widehat{L_2}$, combines to give $\pi_{\lambda_1+\lambda_2} \in \widehat{N_2}$ with coefficients $|f_{u,v}| \in L^2(N_2/Z_1 Z_2)$,
- In fact $||f_{u,v}||^2_{L^2(N_2/Z_1 Z_2)} = \dfrac{||u||^2 ||v||^2}{|P_{\mathfrak{l}_1}(\lambda_1) P_{\mathfrak{l}_2}(\lambda_2)|}$.
- Iterate the construction: $\lambda_r \in \mathfrak{z}_r^*$ with each $P_{\mathfrak{l}_r}(\lambda_r) \neq 0$, and the square integrable $\pi_{\lambda_r} \in \widehat{L_r}$, combine to give $\pi_\lambda \in \widehat{N}$ with coefficients $|f_{u,v}| \in L^2(N/Z_1 \ldots Z_m)$, in fact
- $||f_{u,v}||^2_{L^2(N/Z_1 \ldots Z_m)} = \dfrac{||u||^2 ||v||^2}{|P_{\mathfrak{l}_1}(\lambda_1) \ldots P_{\mathfrak{l}_m}(\lambda_m)|}$.

Reformulate

- $S = Z_1 Z_2 \dots Z_m$ has Lie algebra $\mathfrak{s} = \mathfrak{z}_1 + \mathfrak{z}_2 + \cdots + \mathfrak{z}_m$ so $\mathfrak{s}^* = \mathfrak{z}_1^* + \mathfrak{z}_2^* + \cdots + \mathfrak{z}_m^*$
- $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_m$ with $\lambda_r \in \mathfrak{z}_r^*$
- view b_λ as an antisymmetric bilinear form on $\mathfrak{n}/\mathfrak{s}$
- $P(\lambda) = \mathrm{Pf}(b_\lambda) = P_{\mathfrak{l}_1}(\lambda_1) P_{\mathfrak{l}_2}(\lambda_2) \dots P_{\mathfrak{l}_m}(\lambda_m)$
- If $P(\lambda) \neq 0$ then $\pi_\lambda \in \widehat{N}$ has coefficients $|f_{u,v}| \in L^2(N/S)$
- $||f_{u,v}||^2_{L^2(N/S} = \frac{||u||^2 ||v||^2}{|P(\lambda)|}$
- These representations π_λ are the *stepwise square integrable* representations of N.

Plancherel Measure & Fourier Inversion

- π_λ has distribution character

$$\Theta_\lambda(f) = \mathrm{trace} \int_N f(g)\pi_\lambda(g)dg \text{ for } f \in \mathcal{S}(N)$$

- Plancherel measure on $\widehat{N}$ is concentrated on $\{\lambda \in \mathfrak{s}^* \mid P(\lambda) \neq 0\}$ and given by $(const)|P(\lambda)|d\lambda$
- Fourier inversion formula

$$f(x) = (const) \int_{\mathfrak{s}^*} \Theta_\lambda(r_x f)|P(\lambda)|d\lambda \text{ for } f \in \mathcal{S}(N)$$

Compact Quotients

- N: nilpotent Lie group with stepwise square integrable representations
- Γ: discrete subgroup with N/Γ compact in a way that is consistent with the decomposition $N = L_1 L_2 \dots L_m$:
- $\Gamma \cap N_r$ cocompact in $N_r = L_1 L_2 \dots L_r$ for $1 \leqq r \leqq m$
- $L^2(N/\Gamma) = \sum_{\pi \in \widehat{N}} mult(\pi)\pi$ discrete direct sum with multiplicities $mult(\pi) < \infty$
- $mult(\pi) > 0$ only for $\pi = \pi_\lambda$ with λ integral in the sense that $\exp(2\pi i \lambda)$ is well defined on the torus $Z/(\Gamma \cap Z)$
- Theorem. Let $\lambda \in \mathfrak{s}^*$ with $P(\lambda) \neq 0$, i.e. with π_λ stepwise square integrable. Then (with appropriate normalizations of measures) the multiplicity $m(\pi_\lambda) = |P(\lambda)|$.

Iwasawa Decomposition

- G real reductive Lie group, $G = KAN$ Iwasawa decomp
- N maximal unipotent subgroup
- Theorem. N satisfies the conditions for stepwise square integrable representations

  $N = L_1 L_2 \dots L_{m-1} L_m$ where (a) each L_r has unitary reps with coef. in $L^2(L_r/Z_r)$,

  (b) each $N_r := L_1 L_2 \dots L_r$ is a normal subgp of N with $N_r = N_{r-1} \rtimes L_r$ semidirect,

  (c) $\mathfrak{l}_r = \mathfrak{z}_r + \mathfrak{v}_r$, $\mathfrak{v} = \oplus \mathfrak{v}_r$, $[\mathfrak{l}_r, \mathfrak{z}_s] = 0$ and $[\mathfrak{l}_r, \mathfrak{l}_s] \subset \mathfrak{v}$ for $r > s$.

- Idea of proof – at least the construction:
  - $\{\beta_1, \dots, \beta_m\}$ maximal set of strongly orthogonal $\mathfrak{a}$–roots (cascade down)
  - $\Delta_1^+ = \{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \mid \beta_1 - \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})$
  - $\Delta_{r+1}^+ = \{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a}) \setminus (\Delta_1^+ \cup \dots \cup \Delta_r^+) \mid \beta_{r+1} - \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})\}$
  - $\mathfrak{l}_r = \mathfrak{g}_{\beta_r} + \sum_{\Delta_r^+} \mathfrak{g}_\alpha$ for $1 \leqq r \leqq m$
- Upper triangular matrices: case $G = GL(n; \mathbb{R})$ or $SL(n; \mathbb{R})$

Minimal Parabolics I

- $P = MAN$: minimal parabolic subgroup of G, $M = Z_K(A)$
- principal M-orbits on $\mathfrak{s}^*$: $\mathrm{Ad}^*(M)\lambda$ where $P(\lambda) \neq 0$
- have measurable choice of base points λ^b for principal orbits $\mathrm{Ad}^*(M)\lambda$ with all isotropy subgroups the same
- a polynomial on $\mathfrak{s}^*$, defined by Pf, transforms by the modular function of P, and its Fourier transform D is a differential operator on P (or on AN) that balances lack of unimodularity in the Plancherel formula
- for $a \in A$, $\mathrm{Ad}(a)\mathrm{Det}_{\mathfrak{s}^*} = \left(\prod_r \exp(\beta_r(\log a))^{\dim \mathfrak{z}_r}\right) \mathrm{Det}_{\mathfrak{s}^*}$
- D is an invertible self–adjoint diff op of degree $\frac{1}{2}(\dim \mathfrak{n} + \dim \mathfrak{s})$ on $L^2(MAN)$ with dense domain $\mathcal{C}(MAN)$, and $f(x) = \int_{\widehat{P}} \mathrm{trace}\ \pi(D(r(x)f))d\mu_P(\pi)$

Minimal Parabolics II

- Write $\mathfrak{u}^*$ for the nonsingular set $\{P(\lambda) \neq 0\}$ in $\mathfrak{s}^*$.
- Choose points in $\mathfrak{u}^*$ where isotropies $M_\diamond, A_\diamond, (MA)_\diamond = M_\diamond A_\diamond$ same for each orbit
- $M_\diamond = FM_\diamond^0$ where $F = \exp(i\mathfrak{a}) \cap K$ trivial on $\mathfrak{s}^*$, $M = FM^0$
- Stepwise sq-int π_λ extends to rep $\pi_\lambda^\dagger$ of $M_\diamond$ on $\mathcal{H}_{\pi_\lambda}$
- if $\gamma \in \widehat{M_\diamond}$ set $\eta_{\lambda,\gamma} = \mathrm{Ind}\,_{NM_\diamond}^{NM}(\pi_\lambda \otimes \gamma)$
- and if $\phi \in \mathfrak{a}_\diamond$ set $\pi_{\lambda,\gamma,\phi} = \mathrm{Ind}\,_{NA_\diamond M_\diamond}^{NAM}(\pi_\lambda \otimes e^{i\phi} \otimes \gamma)$
- $\{\mathcal{O}_1, \ldots, \mathcal{O}_v\}$: the (open) $\mathrm{Ad}^*(MA)$–orbits on $\mathfrak{u}^*$; $\lambda_i \in \mathcal{O}_i$
- Characters $\Theta_{\pi_{\lambda,\gamma,\phi}}$ are tempered; if $f \in \mathcal{S}(MAN)$ then
- $f(x) = c\sum_{\{\mathcal{O}_i\}} \sum_{\widehat{M_\diamond}} \int_{\mathfrak{a}_\diamond^*} \Theta_{\pi_{\lambda_i,\gamma,\phi}} D(r(x)f)|P(\lambda_i)|\dim\gamma\, d\phi$

Non-Minimal Parabolics

- The real parabolics containing P are parameterized by subsets $\Phi \subset \Psi$ of the simple restricted root system
- Denote $Q_\Phi = M_\Phi A_\Phi N_\Phi$
- Add together the $\mathfrak{l}_i \cap \mathfrak{n}_\Phi$ for the same $\beta_i|_{\mathfrak{a}_\Phi}$: $\mathfrak{n}_\Phi = \sum_j \mathfrak{l}_{\Phi,j}$
- Then $N_\Phi = L_{\Phi,1} L_{\Phi,2} \dots L_{\Phi,\ell}$ has stepwise square integrable representations – with a slight weakening of one of the technical conditions
- The Dixmier–Pukánszky operator D is similar to the minimal parabolic case: Fourier inversion for $A_\Phi N_\Phi$
- Extension to the parabolic $M_\Phi A_\Phi N_\Phi$ is not yet settled: the problem is how to fit the the $\mathfrak{a}_\Phi$–weight spaces on $\mathfrak{n}_\Phi$ together with the $\mathfrak{l}_{\Phi,j}$, for example whether the $\beta_i|_{\mathfrak{a}_\Phi}$-weight space is contained in an $\mathfrak{l}_{\Phi,j}$

Infinite Dimensional Groups

- G: finitary simple ∞–dim real reductive Lie group
- $G = \varinjlim G_n$ where
  - (i) the restricted root Dynkin diagram $\mathcal{D}_{G_n}$ is a subdiagram of $\mathcal{D}_{G_{n+1}}$
  - (ii) the ordered set $\{\beta_1, \dots, \beta_{m_n}\}$ of strongly orthogonal roots restricted roots for G_n extends to $\{\beta_1, \dots, \beta_{m_{n+1}}\}$
- Example: $G = SL(\infty; \mathbb{H})$, $\ell > 0$ and $G_n = SL(2\ell + 4n; \mathbb{H})$
- Nilradicals of minimal parabolics have decompositions $N_n = L_1 L_2 \dots L_{m_n}$ with N_n normal in N_{n+1}, Mackey obstructions vanish so $\pi_{\lambda_1 + \dots + \lambda_{m_n}}$ extends from N_n to N_{n+1} and we construct stepwise square integrable representations $\pi_{\lambda_1 + \dots + \lambda_{m_n}}$ of N_{n+1}.
- This constructs stepwise square integrable unitary representations $\pi_\lambda = \varinjlim \pi_{\lambda_1 + \dots + \lambda_{m_n}}$ of $N := \varinjlim N_n$

Happy Birthday Becky!!