Winning Strategies: A Mathematical Approach to Games

January 16, 2018

Instructions: Introduce yourself to your neighbors, if you have not already done so. All the games in this worksheet are two-players games so pair up before starting a game and change pairs when you move to the next game. Read the descriptions of the games carefully. First try to play the game a few times, then you can start discussing a solution strategy with your neighbors. I may ask you or your colleagues to share your findings (and possibly write them on the board). If your group finishes early, you can move on to the next game.

1 Counting back from 15

Between you and your partner decide who will go first in the game. Starting from 15, each player in her turn can take away either one or two from the previous number.

1.1 Version A

In this version the first player to reach 0 wins.

(i) Take turns in being the first or the second player. Who do you think is more likely to win the game?

(ii) What strategy should a player use?

(iii) Can you describe a strategy such that one of the two players is *guaranteed* to win?

1.2 Version B

In this version the player who is forced to reach 0 loses.

(i) Take turns in being the first or the second player. Who do you think is more likely to win the game?

(ii) What strategy should a player use?

(iii) Can you describe a strategy such that one of the two players is *guaranteed* to win?

(iv) How does version A compare to version B?

2 Red and black cards

There are 6 red cards and 6 black cards on the table, uncovered. In each turn, a player has to pick two cards. If the cards picked are of the same color, they get replaced with a red card. If they are of different colors, they get replaced by a black card. The first player wins if the last card is black, the second player wins if the last card is red.

(i) Take turns in being the first or the second player. Who do you think is more likely to win the game?

(ii) What strategy should a player use?

(iii) Can you describe a strategy such that one of the two players is *guaranteed* to win?

(iv) Can you find a mathematical model that explains why your solution is right?

Challenge: Suppose now that we change the number of cards on the table. Does this change the game? Who wins in which cases?

3 Poison cookie

Between you and your partner decide who will go first in the game. Starting with a 6x6 grid, at each turn a player picks a box. Then that box and all the boxes above and to its right get eliminated from the game. The bottom left box contains a cookie, but the cookie is poisoned so the player who is forced to eat the cookie (picks the bottom left box) loses the game.

(i) Take turns in being the first or the second player. Who do you think is more likely to win the game?

(ii) What strategy should a player use?

(iii) Can you describe a strategy such that one of the two players is *guaranteed* to win?

(iv) Suppose now that the cookie is a great chocolate cookie so the player who eats the cookie (picks the bottom left box) wins the game. Does this change the game? Who wins?

Challenge: Suppose now that we change the size of the playing grid. Does this change the game? Who wins in which cases?

4 Dominoes on a grid

Between you and your partner decide who will go first in the game. Staring with a 6x6 grid, at each turn a player places down a domino (a 1x2 or 2x1 rectangle). Dominoes cannot overlap, and the first person who cannot place a domino down loses.

(i) Take turns in being the first or the second player. Who do you think is more likely to win the game?

(ii) What strategy should a player use?

(iii) Can you describe a strategy such that one of the two players is *guaranteed* to win?

Challenge: Suppose now that we change the size of the playing grid. Does this change the game? Who wins in which cases?

5 Nim

There are two piles of coins of 6 coins on the table. Each player in her turn chooses only pile and she has to pick at least one coin, but she can pick any number of coins she wishes. The player who takes the last coin wins.

(i) Take turns in being the first or the second player. Who do you think is more likely to win the game?

(ii) What strategy should a player use?

(iii) Can you describe a strategy such that one of the two players is *guaranteed* to win?

(iv) Can you find a mathematical model that explains why your solution is right?

Challenge 1: Suppose now that we change the number of coins in each pile. Does this change the game? Who wins in which cases?
Challenge 2: Suppose now that we have a different number of coins in the two piles. Does this change the game? Who wins in which cases?
Challenge 3: Suppose now that we change the number of piles. Does this change the game? Who wins in which cases?

6 Cutthoroat

6.1 Rules

This game is played on a *graph* (i.e., a set of dots connected by lines). In her turn, a player picks a vertex (i.e., a dot), deletes it and she deletes all the edges (i.e., lines) starting from that vertex. The player who is left with only dots and no lines loses.

6.2 Playing on stars

An *n-star graph* is a central vertex with n edges coming out of it, each ending in one vertex.

(i) Draw a 5-star graph.

(ii) Play cutthroat on a 5-star graph. Who wins the game?

(iii) Suppose that now you have two 5-stars. Does the game change? Who wins?

(iv) Draw a 6-star graph.

(v) Play cutthroat on a 6-star graph. Who wins the game?

(vi) Suppose that now you have two 6-stars. Does the game change? Who wins?

(vii) Suppose that now you have one 5-star and one 6-stars. Does the game change? Who wins?

(viii) Suppose that you have one n-star and one m-star. Can you determine the conditions on n and m that cause one or the other player to win?

(ix) Suppose that you have one n-star and one m-star. Describe a strategy such that one of the two players is *guaranteed* to win.

Challenge 1: What about three stars? Or four stars?
Challenge 2: Try playing cuttrhoat on a Petersen graph. Who wins?

6.3 Line graphs

An *n-line graph* is one line of n vertices each connected to the next by one edge.

(i) Draw a 3-line graph.

(ii) Play cutthroat on a 3-line graph. Who wins the game?

(iii) Suppose that now you have two 3-line graphs. Does the game change? Who wins?

(iv) Draw a 4-line graph.

(v) Play cutthroat on a 4-line graph. Who wins the game?

(vi) Suppose that now you have one 5-line graph. Does the game change? Who wins?

(vii) Make a conjecture about who wins in an n-line graph.

(viii) Test your conjecture by playing cutthroat on a 6-line graph.

Challenge 1: Try playing on a 7-path graph. Keep all your previous findings handy as this will help!
Challenge 2: Could you think of a systematic way to solve the problem of an n-path graph?

7 Some background

All the games in this worksheet are *combinatorial games.* Combinatorial games have the following features:

- they are *two-player* games,
- they are *finite* meaning they always end,
- they are *impartial* as both players have the same options for a move,
- they are *perfect information* games, so there is no hidden information,
- there *not* games of *chance*,
- *no ties* are allowed (usually the first player who cannot make a move loses).

In this situation, it has been proven that a winning strategy exists (either for the first player or the second player). The fact that a winning strategy exists does not men it is easy to find! Finding the winning strategy may need an exhaustive search using computers. For example no winning strategy is known in Hex, another combinatorial game. A surprising result of Sprague and Grundy is that every game with all the features listed above is equivalent to some version of the game of Nim.

7.1 N or P games

A game where the first player has a winning strategy is called an N game, meaning that the *next* player to make a move will win. If there is a winning strategy for the second player, the game is called a P game. The name comes from the following scenario: suppose that we were playing a game and we come to a position where the next player will lose, no matter what she does. Then the *previous* player has won, so she had a winning strategy. If we started the game from this particular position, it would be a second player (or previous player) win, so a P game.

Note that the relationship between P and N positions is not obvious. It is easy to determine if a position is an N position: if it is my turn and I have some way to make a move and put the game into a P position, then I am guaranteed to win (as I become the previous player in a P position and the next player is guaranteed to lose). On the other hand, a game is in a P position only if no matter what move I make, I am forced to put the game into a N position, so I am guaranteed to lose (as I become the previous player in a N position where the next player is guaranteed to win). It is sometimes hard to reason this through at every stage so we can just remember that: the game is N if there is a move to P, the game is P if every move leads to N.

7.2 Sum of games

When a game is the result of the *sum* of more games, meaning that at each turn a player can choose to make a move in any one of the games available, we can use what we know about the smaller games to decide whether in the overall game there is a winning strategy for the first or the second player. It turns out that:

- P + P = P,
- P + N = N,
- N + N = undetermined,

meaning that the sum of two N games may be P or N, we cannot tell a priori. We often use the sum of games rules in cuttrhoat to determine if a game is N or P as there are several moves where we can break a graph into two parts (i.e., we disconnect the graph) and then we can study the two parts as separate games.
Challenge: See if you can justify the rules above!

7.3 Symmetry

Another interesting property is the following. Any game which is the sum of two identical games is a P game. Think about this for a second... How can we prove this?

The winning strategy is remarkably simple: the second player just needs to copy what the first player does! So if player one makes one move on the first copy of the game, the second player does the same thing on the second copy. Then we are back to a game which is the sum of two identical games. Repeat until the second player makes the last move, winning the overall game.
Challenge: Use this to prove that every line graph with an odd number of vertices is an N game.

