The Travels of Two

Amy Shell-Gellasch

Egypt, 2014

You and a friend have been sightseeing in Cairo all morning. You just finished eating at a great falafel place near the bazaar. Your bill is 120 LE (Egyptian pounds). You think the service was just OK, so you want to tip 15 percent. But your friend thought the service was great and wants to tip 20 percent. How would you figure out either tip quickly in your head? (Go ahead, I’ll wait . . .)

If you calculated the 15 percent tip by taking one and a half times 12 LE to get 18 LE, and your friend doubled 12 LE to get 24 LE, you are both in good, as well as ancient, company.

Many cultures across the world and through the ages have used doubling and halving to multiply and divide. In fact, many children inherently double when multiplying, especially when multiplying by powers of two. When my son was 4 years old, he computed $8 \times 10$ by doubling 10, then doubling that, and then doubled that, to get 80—this was before he learned to append a trailing zero. (Being a college math teacher, I was overjoyed. Alas, now that he is learning multiplication in school, he tries to memorize or count on his fingers and no longer uses those clever tricks.)

Ethiopia, 1930

Now you and your friend are hiking through Ethiopia and come to a rural village. Did I mention that your travels might also take you back in time? You see a farmer in the market area who wants to buy 23 baskets from a basket weaver. The baskets are priced at nine shells each. Neither of the men knows the total price so they go to the village shaman.

The shaman makes two columns of small holes in the ground. In the first column, he puts 23, 11, 5, 2, and 1 pebbles. In the second column he puts 9, 18, 36, 72, and 144 pebbles. Not having a dirt floor, we put them in a mancala board, one of numerous games using pebbles and holes from the Middle East and Africa (see figure 1). In his head the shaman designates each row as good or evil. Quickly he says that the baskets cost 207 shells. Not too bad if you have traded your calculator for pebbles or had never heard of a calculator. If you think you know the shaman’s technique, try it on your own.

Russia, 1840

You continue your travels and find yourself deep in Russia, once again in a small village on market day. A Russian farmer is selling 28 bushels of grain for 49 kopeks each to the local cow herder for winter feed. On a scrap of paper or on a scratchboard, he writes two rows of numbers (see figure 2). Off to the side he performs a quick sum and declares the total is 1,372 kopeks.

What is his method? The same as the shaman’s? Grab paper and a pencil—or, as would have been done in an early time, a slate and some chalk; or some
parchment and a quill; or a wax board and stylus; or some papyrus and a reed pen; or a clay tablet and a stylus; or a stick and a patch of smooth dirt—and try to figure it out. I have some grading to do, so take your time . . .

The methods are basically the same. They have been around for hundreds of years, if not millennia, and are known as the Ethiopian and the Russian peasant methods, respectively. The farmer and the shaman start with the first multiplier, 49 in this case, and then start halving, rounding down whenever there is a remainder, until reaching one. Beneath or next to those numbers, they write the second multiplier, in this case 28, and repeatedly double it until the lists have the same number of entries. The shaman designates the odd numbers in the first list as “good” and the even ones “evil.” He adds the numbers paired with the good numbers to get the answer. The farmer does the same, albeit without the colorful designations.

Thus, \( 28 \times 49 = 28 + 448 + 896 = 1,372 \) kopeks, and \( 9 \times 23 = 9 + 18 + 36 + 144 = 207 \) shells.

In a nutshell (or seashell, as the case may be): We convert the larger number into binary and then multiply by the smaller number. When we convert to binary, each division of an odd number by two leaves a remainder of 1 and produces a 1 in the binary representation. This explains why we drop the remainder (round down) when we halve an odd number.

Here is the Russian example worked out using the Euclidian algorithm for finding the binary representation of the number.

\[
49 = 2(24) + 1 \quad \text{odd/good}
\]

\[
= 2(12) + 1 \quad \text{even/evil}
\]

\[
= 2(6) + 1 \quad \text{even/evil}
\]

\[
= 2(3) + 1 \quad \text{odd/good}
\]

\[
= 2(1) + 1 \quad \text{odd/good}
\]

In other words, \( 49 = 2^5 + 2^4 + 1 \), or 110001 in binary. So

\[
49 \times 28 = 2^5(28) + 2^4(28) + 1(28)
\]

\[
= 896 + 448 + 28 = 1,372
\]

This analysis shows why the doubled quantities are aligned with the halved quantities. The juxtaposition allows the user to see immediately which doubled quantities correspond to the odd, or “good,” halved quantities and should be retained. Those are exactly the quantities that would be multiplied by a 1 in the binary representation of the multiplier.

No records of the Ethiopian or Russian methods exist because they were not written down (the Ethiopian method) or were only scratched out on whatever writing surface was available (the Russian method). But for several hundred years, and in many areas, travelers have reported seeing these methods used. So we can assume that they have been used for much longer and that they would have appeared quite naturally in almost any culture that evolved to the point of needing to multiply and divide.

In fact, we have evidence, in the form of preserved papyri and engravings, that the ancient Egyptians developed methods of computation that relied on doubling and halving.

**Egypt, 1550 BCE**

You and your friend resume your travels in time and space once more, landing in ancient Egypt. You happen upon an Egyptian scribe seated cross-legged with papyrus on his lap. He is computing the same product that our previous merchants performed. What a coincidence! The scribe is writing in a script foreign to you, but it looks like mathematics. His calculation of \( 49 \times 28 \), written in modern Indo-Arabic numerals, is shown in figure 3.

The marks on the left are the key to the solution. How are they chosen? Yep, \( 4 + 8 + 16 = 28 \). Just as with the other methods, one of the multipliers—28 in this case—is in essence converted to base two. By summing the multiples of the other multiplier, 49, that correspond to the powers of two in the binary representation.
tion, we obtain the product. Because all numbers can be uniquely represented in binary, this method provides an accurate and quick way to perform multiplication.

But the Egyptians did not stop there. They adapted this process to perform division. To calculate $1,372 \div 28$, we start the same way: Repeatedly double 1 and the divisor 28 (see figure 4). However, now we look for multiples of the divisor that add to the dividend. As soon as a multiple of the divisor exceeds half the dividend, we can stop because the next one will be too large. The multiples of 28 that add to 1,372 are marked with a backslash. So the quotient is $1 + 16 + 32 = 49$.

Amazingly, the Egyptians could use this doubling method and its reverse (halving) to multiply and divide any combination of whole numbers and fractions, including approximating the quotient when the division is not exact. They were even able to solve linear equations.

The Egyptian methods of performing arithmetic with fractions are quite complex, in part because the Egyptians represented fractions as sums of unit fractions (and the lone exception, $2/3$). Their rules for representing fractions adhered to four basic guidelines, one of which is that no fraction can be repeated. For example, $9/10 = 2/3 + 1/5 + 1/30$, so the Egyptians would have written it as the characters for $2/3$ $1/5$ $1/30$, with addition assumed (see figure 5). For simplicity, we will use western fractions.

We illustrate by multiplying a whole number and a fraction: 9 and $1/5$ (see figure 6). As before, we start with 1 on the left and the fractional factor on the right. We repeatedly double the entries in the two columns. Doubling a unit fraction is easy if the denominator is even, but not as easy when it is odd. To do this, the ancient scribe consulted tables that list the doubles of such fractions, referred to now as $2/N$ (“2 over $N$”) tables. For example, the Mathematical Leather Roll contains a $2/N$ table. It was purchased with the Ahmes (Rhind)

\[
\begin{array}{cccc}
\hline
\text{Row} & \text{2} & \text{56} & \text{112} & \text{224} \\
\hline
\text{16} & 448 & \hline
\text{32} & 896 & \hline
\end{array}
\]

Figure 4. The Egyptian method of computing $1,372 \div 28$.

\[
\begin{array}{cccc}
\hline
\text{Row} & \text{1} & \text{1/3} & \text{1/5} & \text{1/15} \\
\hline
\text{2} & 2 & \text{1/3} & \text{1/15} & \hline
\text{4} & \text{2/3} & \text{1/10} & \text{1/30} & \hline
\text{8} & 1 & \text{1/3} & \text{1/5} & \text{1/15} & \hline
\text{9} & 1 & \text{1/3} & \text{1/5} & \text{1/15} & \text{1/15} & \hline
\text{1} & \text{2/3} & \text{1/10} & \text{1/30} & \hline
\end{array}
\]

Figure 6. The Egyptian method of computing $9 \times (1/5)$.

Mathematical Papyrus—a famous papyrus that contains numerous examples of the doubling method.

A $2/N$ table states that the double of $1/5$ is $1/3$ $1/15$ (i.e., $1/3 + 1/15$); this is used in the second line of figure 6. In the third line, we doubled $1/3$ to get $2/3$, and we consulted a $2/N$ table for the double of $1/15$. We stopped when we found rows on the left that add to 9. The final two lines are simply cleaning up the fractions. For example, $1/5$ $1/5$ is not allowed, so we used the double of $1/5$ again. The final result is $1 + 2/3 + 1/10 + 1/30$, or $9/5$, as expected. Don’t you appreciate our number system much more now?

The Russian and Ethiopian methods of multiplication, Egyptian multiplication and division, and even modern computers, rely on doubling, halving, and binary representations of numbers—though to be historically accurate, other than modern computing, none of these peoples understood the theoretical basis of what they were doing. They would have simply noticed over time that the system worked. They realized that two really has the power!■

Further Reading

For a great introduction to Egyptian mathematics, try Mathematics in the Time of the Pharaohs, by Richard J. Gillings (Mineola, NY: Dover, 1982).

More details can also be found in Algebra in Context, by A. Shell-Gellasch and J. Thoo (Baltimore, MD: Johns Hopkins University Press, 2014).

Amy Shell-Gellasch is a math historian who loves to liberally mix her mathematics and history. She teaches at Montgomery College in Rockville, Maryland, and researches mathematical devices at the Smithsonian’s National Museum of American History.

Email: Amy.Shell-Gellasch@montgomerycollege.edu

http://dx.doi.org/10.4169/mathhorizons.22.1.5

www.maa.org/mathhorizons : Math Horizons : September 2014