There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.
Problem 1

Suppose that $y = h(x, t)$ is a time-dependent ($t$ denotes the time) smooth, invertible mapping from $x \in \mathbb{R}^3$ to $y \in \mathbb{R}^3$ that is also differentiable with respect to $t$. Suppose also that $h(x, 0) = x$.

(a) Let $J(x, t) := \partial y/\partial x$ denote the Jacobian of the mapping at time $t$. Determine the sign of $J$.

(b) Recall that if $M(t)$ is a differentiable time-dependent $N \times N$ matrix, then

$$\frac{d}{dt} \det(M(t)) = \sum_{j=1}^{N} \det(M^{(j)}(t))$$

where $M^{(j)}(t)$ denotes the matrix $M(t)$ with its $j^{th}$ row replaced by its time derivative. (Of course the analogous result also holds where all columns are replaced by their derivatives.) Let

$$v(y, t) := \left. \frac{\partial h}{\partial t}(x, t) \right|_{x = h^{-1}(y, t)}$$

where $x = h^{-1}(y, t)$ denotes the mapping inverse to $y = h(x, t)$. (To be clear, this means: differentiate first with respect to $t$, and then substitute for $x$ as indicated.) Prove that

$$\frac{\partial J}{\partial t}(x, t) = \left. \text{div}_y(v) \right|_{y = h(x, t)} J(x, t).$$

(Note the divergence here is with respect to $y$.)

(c) Suppose that $\Omega_0$ is a bounded domain in $\mathbb{R}^3$ with smooth orientable boundary, and let $\Omega_t$ denote its image under the mapping $h$. Prove Reynold’s Transport Theorem (generalizing Leibniz’ rule):

$$\frac{d}{dt} \int \int \int_{\Omega_t} f(y, t) dV(y) = \int \int \int_{\Omega_t} \left[ \frac{\partial f}{\partial t} + \text{div}_y(fv) \right] dV(y),$$

where $f(y, t)$ is an arbitrary smooth function of its arguments and where the vector field $v$ is defined as in part (b).
Problem 1
Problem 1
Problem 1
Problem 2

By considering a suitable semicircular contour in the upper half-plane and the analytic function \( z^{1/2} \log(z)/(1 + z^2) \) (\( z^{1/2} \) and \( \log(z) \) denote principal branches), evaluate by one and the same calculation the two integrals

\[
I := \int_0^\infty \frac{\sqrt{x}}{1 + x^2} \, dx \quad \text{and} \quad J := \int_0^\infty \frac{\sqrt{x} \log(x)}{1 + x^2} \, dx.
\]
Problem 2
Problem 2
Problem 2
Problem 3

Consider the infinite series

\[ S(x) := \sum_{n=0}^{\infty} a_n \sin(nx), \quad x \in \mathbb{R} \]

under the assumption that

\[ \sum_{n=0}^{\infty} |a_n| < \infty. \]

Let \( S_M(x) \) denote the corresponding partial sum:

\[ S_M(x) := \sum_{n=0}^{M} a_n \sin(nx). \]

(a) Given an arbitrary \( \epsilon > 0 \), show that there exists \( N(\epsilon) \) such that \( |S(x) - S_M(x)| < \epsilon \) for all \( M > N(\epsilon) \) and all \( x \in \mathbb{R} \), thereby establishing the uniform convergence of the series.

(b) Prove that

\[ |S_M(x + h) - S_M(x)| \leq h \sum_{n=0}^{M} n|a_n|. \]

(c) Prove that \( S(x) \) is a uniformly continuous function of \( x \in \mathbb{R} \).
Problem 3
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Problem 3
**Problem 4**

The goal of this problem is to prove the following Theorem due to Weierstraß:

**Theorem 1** Suppose $f(z)$ is analytic for $z$ in a punctured neighborhood of $z_0 \in \mathbb{C}$ (that is, for $z \neq z_0$ but $|z - z_0|$ sufficiently small). Suppose also that $f(z)$ has an essential singularity at $z_0$. Then for each $w \in \mathbb{C}$, $\epsilon > 0$, and $\delta > 0$, there exists some $z$ with $|z - z_0| < \delta$ for which $|f(z) - w| < \epsilon$. In other words, $f$ takes values arbitrarily close to any given complex number in a neighborhood of an essential singularity.

To this end, suppose to the contrary that for some $w \in \mathbb{C}$, $\epsilon > 0$, and $\delta > 0$ we have $|f(z) - w| \geq \epsilon$ for all $z$ with $|z - z_0| < \delta$. Let

$$h(z) := \frac{1}{f(z) - w}$$

wherever the right-hand side makes sense (in particular we must exclude $z = z_0$).

(a) Show that $h(z)$ is analytic for $0 < |z - z_0| < \delta$.

(b) Represent $h(z)$ by its Laurent series about $z = z_0$, and prove directly that all of the coefficients of $(z - z_0)^n$ for $n < 0$ vanish.

(c) In light of the calculation from part (b), show how to define $h(z)$ for $z = z_0$ so that $h(z)$ is analytic for $|z - z_0| < \delta$.

(d) Express $f(z)$ in terms of $h(z)$ and complete the proof (by contradiction) of Weierstraß’s theorem.
Problem 4
Problem 4
Problem 4
Problem 5

Let $f(x, y)$ be a vector field in the $(x, y)$-plane that is smooth everywhere except at three points $P_1 = (2, 2)$, $P_2 = (6, 3)$, and $P_3 = (4, 6)$ where $f$ is not defined. Suppose that

$$\text{div}(f(x, y)) = 1, \quad (x, y) \in \mathbb{R}^2 \setminus \{P_1, P_2, P_3\}.$$ 

Assume also that

$$\int_{C_1} f(x, y) \cdot n(x, y) \, ds(x, y) = 3\pi,$$
$$\int_{C_2} f(x, y) \cdot n(x, y) \, ds(x, y) = -\pi,$$
$$\int_{C_3} f(x, y) \cdot n(x, y) \, ds(x, y) = 2\pi,$$

where $C_j$ denotes the circle of radius 1 centered at $P_j$, $n(x, y)$ denotes the unit normal pointing away from the center of each circle, and $ds(x, y)$ denotes the arc length differential.

(a) Let $R$ be the rectangle with vertices (0, 0), (10, 0), (10, 5), and (0, 5). Calculate the flux of $f$ outward through the boundary of $R$.

(b) Give an example of a vector field $f(x, y)$ with the above properties.
Problem 5
Problem 5
Problem 5