1. **Do not open this exam until you are told to do so.**

2. This exam has 11 pages including this cover. There are 11 problems.

3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.

4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.

5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.

6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course.

7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.

8. **Turn off all cell phones and pagers**, and remove all headphones.

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<thead>
<tr>
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1. [8 points]

A function \( f(x) \) is shown above, and graphs of various transformations of \( f(x) \) are shown in the figures below. Circle the one formula beneath each of the following graphs that best describes the transformation(s) of \( f(x) \).

(a)  

![Graph A](image1)

(i) \(-f(kx), k > 1\)
(ii) \(f(-kx), 0 < k < 1\)
(iii) \(f(-kx), k > 1\)
(iv) \(-f(kx), 0 < k < 1\)

(b)  

![Graph B](image2)

(i) \(f(kx), k > 1\)
(ii) \(f(kx), 0 < k < 1\)
(iii) \(kf(x), k > 1\)
(iv) \(kf(x), 0 < k < 1\)

(c)  

![Graph C](image3)

(i) \(af(2x) + b, a > 1, b > 1\)
(ii) \(af(2x) + b, 0 < a < 1, b > 1\)
(iii) \(af(\frac{1}{2}x) + b, a > 1, b > 1\)
(iv) \(af(\frac{1}{2}x) + b, 0 < a < 1, b > 1\)

(d)  

![Graph D](image4)

(i) \(f(x + 4) - 6\)
(ii) \(f(x - 4) - 1\)
(iii) \(f(x + 4) - 1\)
(iv) \(f(x - 4) - 6\)
2. [10 points] For Thanksgiving, Jane is preparing a 15-pound turkey to feed her family. The temperature, $T$, of the turkey $h$ hours after it has been put in the oven is given by $\quad T(h) = 350 - 310e^{kh}$, where $k$ is constant and $T$ is measured in degrees Fahrenheit. Use this information to answer the following questions. Show all work and remember to include units in your answer whenever appropriate.

a. [2 points] What is the initial temperature of the turkey?

**Solution:** Initial temperature is given by $T(0) = 350 - 310e^{k\cdot 0} = 40$.
The initial temperature is 40 degrees Fahrenheit.

b. [2 points] What is the temperature of Jane’s oven when the turkey is being cooked? (Hint: as time increases, the turkey temperature will approach the oven temperature.)

**Solution:** The limit as $h \to \infty$ of $T(h)$ is 350°F.
Hence the oven temperature is 350 degrees Fahrenheit.

c. [4 points] If the temperature of the turkey is 180°F in four hours, evaluate $k$. Be sure to show how you derived your answer. Express your answer exactly; approximate answers will not receive full credit.

**Solution:** We are given that $T(4) = 180$. We use this to solve for $k$.

\[
\begin{align*}
180 &= 350 - 310e^{4k} \\
-170 &= -310e^{4k} \\
\frac{170}{310} &= e^{4k} \\
\ln\left(\frac{170}{310}\right) &= 4k \\
k &= \frac{1}{4} \ln\left(\frac{170}{310}\right)
\end{align*}
\]

$k \approx -0.15019$

d. [2 points] On the other side of town, John is also preparing a 15-pound turkey. He cooks his turkey in an oven that is 375°F. The temperature $J$ of John’s turkey, $h$ hours after it has been in the oven, is given by $\quad J(h) = a - 335e^{-0.155h}$
where $a$ is constant. What is the value of $a$? Explain your reasoning.

**Solution:** As the turkey is in the oven longer, the temperature of the turkey approaches the temperature of the oven. We also know that as $h \to \infty$, $e^{-0.155h} \to 0$, so $J(h) \to a$.
Therefore, $a = 375$. 
3. [9 points] The concessions stand at All-American High School finds that its weekly profit due to popcorn sales at the Friday night football game is a function of the price it charges per bag of popcorn. The weekly profit is given by

\[ W(p) = -100p^2 + 300p - 25, \]

where \( W \) is the profit made in dollars when selling popcorn at the price of \( p \) dollars per bag.

a. [3 points] Evaluate \( W(0) \). In one sentence, explain what the value of \( W(0) \) means in the context of this problem.

\[ Solution: \quad W(0) = -25. \] When the school charges $0 per bag of popcorn, they lose $25.

b. [4 points] Use the method of completing the square to determine how much the concessions should charge per bag of popcorn to earn a maximum profit. Be sure to show all appropriate work, and write your final answer in a complete sentence, including appropriate units.

\[ Solution: \]

\begin{align*}
W &= -100p^2 + 300p - 25 \\
&= -100(p^2 - 3 + 0.25) \\
&= -100(p^2 - 3 + 2.25 - 2.25 + 0.25) \\
&= -100((p^2 - 3 + 2.25) - 2) \\
&= -100(p - 1.5)^2 + 200
\end{align*}

The concessions stand earns a maximum profit of $200 when they charge $1.50 per bag of popcorn.

c. [2 points] Justify why your answer in part (b) gives a maximum.

\[ Solution: \] The vertex gives a maximum because the the coefficient for \( p^2 \) is negative, so the parabola faces downward.
4. [10 points] Katie and Will want to put a fence around a rectangular garden space in their backyard. They will plant squashes and roses in their garden, and there will be a dividing fence between the two types of plants, as shown in the figure below. The dividing fence is parallel to one pair of the sides of the rectangular space. Katie and Will have 120 feet of fencing materials to use. They are not worried about the relative areas of the squash and rose gardens, but they do want to make sure that they maximize the total gardening area. Knowing that they need to provide fencing to surround the garden’s perimeter in addition to the dividing segment, what are the length and width dimensions of the total garden that will yield a maximum area for their garden? What is this maximum area?

Solution: Let L and W be the length and width of the garden, respectively. They have 120 feet of fencing to use along the perimeter of the garden, which is given by $2W + 2L$, in addition to the $W$ length that runs through the middle of the garden. Using this information, we can solve for W in terms of L to find

$$3W + 2L = 120$$
$$W = 40 - \frac{2}{3}L.$$ 

The total area of the garden is $A = LW$. Therefore, we have

$$A = (40 - \frac{2}{3}L)L$$
$$= 40L - \frac{2}{3}L^2$$
$$= -\frac{2}{3}(L^2 - 60L)$$
$$= -\frac{2}{3}(L^2 - 60L + 900 - 900)$$
$$= -\frac{2}{3}(L - 30)^2 + 600.$$ 

When the length is 30 feet, the width must be 20 feet, giving a maximum total garden area of 600 ft$^2$.

Alternate correct solution: Zeros of quadratic $A(L) = (40 - \frac{2}{3}L)L$ are at $L = 0$ and $L = 60$. Since coefficient for $L^2$ is negative, the vertex yields a maximum. By vertical symmetry of a parabola, the input coordinate of the vertex is $L = 30$. Hence the maximal dimensions are 30 feet $\times$ 20 feet, giving a 600 square feet as the maximal area.
5. [8 points] A colony of bacteria grows exponentially in a petri dish. Given that it takes 7 hours for the number of bacteria to double, how long does it take for the colony to triple in size? Analytically solve for the answer and be sure to show all work and use appropriate units in your answer.

Solution: Let $B$ be the amount of bacteria in the colony after $t$ hours. Since the bacteria grows exponentially, we know $B = B_0e^{(kt)}$, where $B_0$ is the initial number of bacteria, and $k$ is the continuous growth rate. When $t = 7$, we know $B = 2B_0$. We use this to solve for $k$:

$$
2B_0 = B_0e^{(kT)} \\
2 = e^{(kT)} \\
\ln(2) = 7k \\
k = \frac{\ln(2)}{7} \approx 0.099021
$$

We now know $B = B_0e^{\frac{1}{T}\ln(2)t}$. We want to find how many hours it takes for $B = 3B_0$.

$$
3B_0 = B_0e^{\frac{1}{T}\ln(2)t} \\
3 = e^{\frac{1}{T}\ln(2)t} \\
\ln(3) = \frac{1}{7}\ln(2)t \\
t = \frac{7\ln(3)}{\ln(2)} \approx 11.095
$$

It takes approximately 11.1 hours for the colony to triple in size.

Alternate solution: Let $B(t) = B_0b^t$ model the growth of the bacteria colony. We are given that $B_0b^7 = 2B_0$, whence $b^7 = 2$, so $b = 2^{1/7} \approx 1.10409$. We seek a time $T$ such that $B_0b^t = 3B_0$:

$$
(2^{1/7})^t = 3 \\
t \log(2^{1/7}) = \log(3) \\
t \left(\frac{1}{7}\right) \log(2) = \log(3) \\
t = \frac{7\log(3)}{\log(2)}
$$

Hence it takes $\frac{7\log(3)}{\log(2)}$, or approximately 11.1, hours.
6. [6 points] Use the data provided for \( f(x) \) to fill in the blank spaces in the following tables, given that the function represented in each table is a transformation of \( f(x) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>5</td>
<td>2</td>
<td>0.5</td>
<td>1.5</td>
<td>-3</td>
<td>-5</td>
<td>-6</td>
</tr>
</tbody>
</table>

a. [2 points]

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(x) )</td>
<td>-4</td>
<td>-7</td>
<td>-8.5</td>
<td></td>
<td>-12</td>
</tr>
</tbody>
</table>

Solution: \( g(x) = f(x) - 9 \), so \( g(0) = -7.5 \).

b. [2 points]

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h(x) )</td>
<td>0.5</td>
<td>-3</td>
<td>-5</td>
<td></td>
<td>-6</td>
</tr>
</tbody>
</table>

Solution: \( h(x) = f(x + 2) \), so \( h(-2) = 1.5 \).

c. [2 points]

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k(x) )</td>
<td>2.5</td>
<td>-1</td>
<td>-3</td>
<td></td>
<td>-4</td>
</tr>
</tbody>
</table>

Solution: \( k(x) = f(x + 2) + 2 \), so \( k(-2) = 3.5 \).

7. [4 points] The function \( j(x) \) is an odd function. Fill in as many values of the table as you can. Put an \( X \) in boxes that you cannot fill in.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( j(x) )</td>
<td>-7</td>
<td>-1</td>
<td></td>
<td></td>
<td></td>
<td>13</td>
<td></td>
</tr>
</tbody>
</table>

Solution: Bottom row should read \(-7, -13, -1, 0, +1, 13, +7\).
8. [15 points] Eric receives an unexpected monetary gift of $3000. Because he took a pre-calculus course in college, Eric knows that investing this money in a good bank account will allow his money to grow exponentially. After visiting several banks in Ann Arbor, he has gathered information to consider various options.

a. [3 points] At Wolverine Bank, his money will be compounded annually at a rate of 4.2%. Find a formula for \( W(t) \) that gives the balance of the account, in dollars, \( t \) years after it is invested.

\[
W(t) = 3000(1.042)^t
\]

Solution:

b. [2 points] At TreeTown Bank, his money will earn interest at a rate of 4.15% per year, compounded monthly. Explain what it means for a balance to be compounded monthly.

Solution: The bank adds interest to the account every month, as opposed to once per year.

c. [3 points] At TreeTown Bank, his money will earn interest at a rate of 4.15% per year, compounded monthly. Find a formula for \( T(t) \) that gives the balance of the account, in dollars, \( t \) years after it is invested.

\[
T(t) = 3000 \left(1 + \frac{0.0415}{12}\right)^{12t}
\]

Solution:

\[
T(t) = 3000 \left(1 + \frac{0.0415}{12}\right)^{12t}
\]

\[
T(t) = 3000 \left(1 + \frac{0.0415}{12}\right)^{12t}
\]

d. [4 points] At Michigan Bank, his money will earn interest at a rate of 4.1% per year, compounded continuously. Find a formula for \( M(t) \) that gives the balance of the account, in dollars, \( t \) years after it is invested. State what the growth factor of this formula is.

\[
M(t) = 3000e^{0.041t}
\]

Solution: The growth factor is \( e^{0.041} \).

\[
M(t) = 3000e^{0.041t}
\]

e. [3 points] Which bank should Eric use to invest his $3000? Justify your answer by comparing the effective annual interest rates offered by each bank.

Solution: Eric should invest in TreeTown Bank because this bank offers the highest effective annual interest rate. The effective annual interest rate at Wolverine Bank is 4.2%. The effective rate at TreeTown Bank can be determined by finding the growth factor, \( (1 + \frac{0.0415}{12})^{12} \approx 1.042298 \), giving an effective annual interest rate of approximately 4.23%. The effective annual interest rate at Michigan Bank can be found by finding the growth factor, \( e^{0.041} \approx 1.041852 \), giving an effective annual interest rate of approximately 4.19%.
9. [8 points] In the imaginary land of Sontik, people live much longer than they do in the real world. As a result, the noble citizens of Sontik determine age, measured in Sontikian eras, with the relationship

\[ S = \log (R + 1) , \]

where \( R \) is the age measured in Earth years and \( S \) is the age measured in Sontikian eras. Use this information to answer the following questions.

a. [2 points] What is the smallest value of \( R \) that can be used within the context of this problem? Briefly justify your answer.

**Solution:** \( R \) can be zero but cannot be negative since it is impossible to have a negative age measured in Earth years.

b. [2 points] Citizen Alpha and Citizen Kane are two residents of Sontik. Let \( S_A \) be the age of Citizen Alpha, and \( S_K \) be the age of Citizen Kane, measured in Sontik eras. The difference between their ages is given by \( D = S_K - S_A \). Find a formula for \( D \) in terms of \( R_A \) and \( R_K \), where \( R_A \) and \( R_K \) are the ages Citizen Alpha and Citizen Kane, respectively, measured in Earth years. Include all the steps of your derivation.

**Solution:**

\[
D = S_K - S_A \\
= \log (R_K + 1) - \log (R_A + 1) \\
= \log \left( \frac{R_K + 1}{R_A + 1} \right)
\]

c. [4 points] Suppose Citizen Kane is 2 Sontikian eras older than Citizen Alpha. Find a formula for Citizen Kane’s Earth age, \( R_K \), as a function of Citizen Alpha’s Earth age, \( R_A \). Include all the steps of your derivation.

**Solution:** From part (b), we have

\[
2 = \log \left( \frac{R_K + 1}{R_A + 1} \right) \\
10^2 = \left( \frac{R_K + 1}{R_A + 1} \right) \\
100(R_A + 1) = R_K + 1 \\
R_K + 1 = 100R_A + 100 \\
R_K = 100R_A + 99
\]
10. [13 points] On steam-powered riverboats, the paddle wheel in the back of the boat rotates in a circular motion to propel the boat through the water. Assume the paddle wheel on the back of the riverboat Huron Queen rotates at a constant speed and suppose a sensor is placed on the very tip of the wheel. Initially, the sensor is at its highest height of 12 feet above the surface of the water. In 5 seconds, the sensor is at its lowest point, which is 5 feet below the surface of the water. Let \( H(t) \) be the height of the sensor above the surface of the water at \( t \) seconds. Positive values of \( H \) denote that the sensor is above the water, whereas negative values of \( H \) denote that the sensor is below the water’s surface.

   a. [7 points] Sketch a graph of \( H(t) \) for the first 20 seconds. Be sure to label the axes appropriately, and denote the midline.

   \[
   \begin{array}{c|cccccccc}
   t \text{ (seconds)} & 0 & 5 & 10 & 15 & 20 \\
   \text{Height, } H \text{ (feet)} & -5 & 0 & 3.5 & 12 & \\
   \text{Time, } t \end{array}
   \]

   \[
   \begin{array}{c|c}
   \text{Time, } t \text{ (seconds)} & 0 & 5 & 10 & 15 & 20 \\
   \text{Height, } H \text{ (feet)} & -5 & 0 & 3.5 & 12 & \\
   \text{Midline} & & & & & \\
   \end{array}
   \]

   \[
   \begin{array}{c|c}
   \text{Solution:} & \\
   \text{Graph of } H(t) & \\
   \end{array}
   \]

   b. [2 points] What is the radius of the paddle wheel?
   \[
   \text{Solution: } 8.5 \text{ feet}
   \]

   c. [2 points] What is the height of the center of the paddle wheel above the water? Justify your answer.
   \[
   \text{Solution: } \text{The height at the center of the wheel is the average height on the wheel, and the midline gives the average value of the function. Since the midline is at 3.5 feet, the center of the wheel is 3.5 above the water.}
   \]

   d. [2 points] How many revolutions does the paddle wheel complete in one minute? Justify your answer.
   \[
   \text{Solution: } \text{It takes 10 seconds for the paddle wheel to complete one full revolution, so it can complete 6 full revolutions in one minute.}
   \]
11. [9 points] Suppose $C(A)$ is the number of cars sold by a local dealership during one month when the dealership spends $A$ dollars on advertising.

a. [3 points] In one complete sentence, interpret $C(500) + 20$ in terms of number of cars and money spent on advertising.

\[ \text{Solution: The expression } C(500) + 20 \text{ is 20 cars more than the number of cars the dealership sells if they spend } \$500 \text{ on advertising.} \]

b. [2 points] To promote employee morale, the manager of the dealership gives all of his employees a $5 bonus for each car the dealership sells. Write an expression for $B(A)$ as a transformation of $C(A)$, where $B(A)$ is the monthly bonus an employee earns when the dealership spends $A$ dollars on advertising.

\[ \text{Solution: } B(A) = 5C(A) \]

c. [2 points] The manager of the dealership finds a new advertising agency that offers to cover the first $200 of advertising for free. Write an expression for $N(A)$ as a transformation of $C(A)$, where $N(A)$ is the number of cars sold when the company spends $A$ dollars on advertising with the new agency.

\[ \text{Solution: } N(A) = C(A + 200) \]

d. [2 points] Due to economic trouble, the dealership finds that its car sales are less than what they used to be. In fact, when the company spends $A$ dollars on advertising, they find that the number of cars they sell in one month is 15 less than what it used to be when they spent $A$ dollars. Find an expression for $E(A)$ as a transformation of $C(A)$, where $E(A)$ is the number of cars sold during the time of economic hardship.

\[ \text{Solution: } E(A) = C(A) - 15 \]