Math 105 — Final Exam
April 23, 2009

Name: ____________________________

Instructor: ____________________________ Section: ____________________________

1. Do not open this exam until you are told to do so.

2. This exam has 11 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, and it may be to your advantage to skip over and come back to a problem on which you are stuck.

3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.

4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.

5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.

6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course.

7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.

8. Turn off all cell phones and pagers, and remove all headphones.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>Score</th>
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1. [10 points] For each statement below, circle TRUE if the statement is \textit{always} true; otherwise, circle FALSE. No partial credit on this page.

For each of the problems on this page, \(a, b,\) and \(k\) are positive constants.

a. [2 points] Suppose the function \(f(x)\) is defined as follows:

\[ f(x) = \frac{ae^x - x^2}{e^{kx} + 2}. \]

Then \(f(x)\) has a horizontal asymptote at \(y = a.\)  

\[ \text{True} \quad \text{False} \]

b. [2 points] \(x^{1/10} < \log(10x)\) for all \(x > 2.\)

\[ \text{True} \quad \text{False} \]

c. [2 points] \(\ln(x + a) + b \ln(x + a) = \ln(x + a)^{(b+1)}\)

\[
\begin{array}{c|c}
\text{True} & \text{False} \\
\end{array}
\]

d. [2 points] If \(m(x)\) is an invertible function, then \(g(x) = am(bx)\) is also an invertible function.

\[
\begin{array}{c|c}
\text{True} & \text{False} \\
\end{array}
\]

e. [2 points] If \(n(x)\) is an even function, then \(h(x) = n(x + a)\) cannot be an even function.

\[
\begin{array}{c|c}
\text{True} & \text{False} \\
\end{array}
\]
2. [8 points] For each of the following scenarios, find a formula that mathematically models the situation. You do not need to show your work for this page.

a. [2 points] Chelsea invests $700 in an account that earns 8.2% interest annually. Find a formula for $C(t)$, the amount of money Chelsea has in her account $t$ years after she originally invested.

\[
C(t) = 700(1.082)^t
\]

b. [2 points] When first planted in Amy’s yard, a maple tree was measured to be 6 feet tall. The tree then grew 3 inches per year in height. Find a formula for $h(t)$, the height of the tree $t$ years after planted in Amy’s yard, where $h(t)$ is measured in feet.

\[
h(t) = 6 + 0.25t
\]

c. [2 points] A certain creature’s weight is proportional to the square of its height. If the creature weighs 1.2 pounds when it is 4.0 inches tall, find a formula for $w(t)$, the creature’s weight, in pounds, when its height is $t$ inches.

\[
w(t) = 0.075t^2
\]

d. [2 points] The mass of a radioactive substance decays at a continuous rate of 4.6% per year. There were 300 grams of the substance in 1980. Find a formula for $S(t)$, the grams of the substance remaining $t$ years after 1980.

\[
S(t) = 300e^{-0.046t}
\]
3. [5 points] In business, a company’s monthly profit is determined by its monthly revenue minus its monthly costs. The figure below shows the monthly revenue curve, $R(x)$, and the monthly cost curve, $C(x)$, for Company ABC when producing and selling $x$ number of goods. On the same axes, sketch the graph of the monthly profit curve, $P(x)$, for Company ABC when producing and selling $x$ goods.
4. [10 points] Recall that a company’s monthly profit is determined by its monthly revenue minus its monthly costs. Company XYZ sells shoes. Each pair of shoes is sold for $70.

a. [2 points] Find an equation for $R(x)$, the monthly revenue in dollars generated for Company XYZ when selling $x$ pairs of shoes in a month.

Solution:

$$R(x) = 70x$$

b. [2 points] The monthly cost in dollars of producing $x$ shoes for Company XYZ is given by $C(x) = 0.4x^2 - 50x + 3000$. Using your answer from part (a), find an equation for $P(x)$, the monthly profit of Company XYZ when selling and producing $x$ pairs of shoes per month.

Solution:

$$P(x) = R(x) - C(x) = 70x - (0.4x^2 - 50x + 3000) = -0.4x^2 + 120x - 3000$$

c. [6 points] Using your answer from part (b) and by completing the square for $P(x)$, find how many pairs of shoes Company XYZ should produce and sell to have a maximum monthly profit. What is this maximum profit? Give your final answers in a complete sentence.

Solution:

$$P(x) = -0.4x^2 + 120x - 3000 = -0.4 \left[ x^2 - 300x + 3000 \right] = -0.4 \left[ \left( x - 150 \right)^2 - 15000 \right] = -0.4(x - 150)^2 + 6000$$

The vertex of the quadratic profit function is at $(150, 6000)$, which we know is a maximum because the parabola opens downward. Per month, Company XYZ should produce and sell 150 pairs of shoes to earn a maximum monthly profit of $6000.$
5. [12 points] A national team of scientists originally is composed of 16 women and 84 men. However, the committee forming this team is concerned about the lack of female representation and would like to add women to the team without removing any of the original male or female members.

a. [3 points] Let \( f(w) \) be the fraction of the team that is female when \( w \) women have been added. Find a formula for \( f(w) \).

\[
\text{Solution:} \\
f(w) = \frac{16 + w}{100 + w}
\]

b. [3 points] In one complete sentence, evaluate and interpret \( f(5) \).

\[
\text{Solution: } f(5) \text{ is the fraction of the team that is female when 5 women are added, which is } \frac{21}{105}, \text{ or } \frac{1}{5}.
\]

c. [3 points] Find a formula for \( f^{-1}(w) \).

\[
\text{Solution: } \text{Let } y = f(w). \text{ Then we have}
\]

\[
\begin{align*}
y &= \frac{16 + w}{100 + w} \\
y(100 + w) &= 16 + w \\
100y + yw &= 16 + w \\
yw - w &= 16 - 100y \\
w(y - 1) &= 16 - 100y \\
w &= \frac{16 - 100y}{y - 1} \\
&= \frac{100y - 16}{1 - y}
\end{align*}
\]

So \( w = f^{-1}(y) = \frac{100y - 16}{1 - y} \), which means \( f^{-1}(w) = \frac{100w - 16}{1 - w} \).

d. [3 points] In one complete sentence, evaluate and interpret \( f^{-1}(0.4) \).

\[
\text{Solution: } f^{-1}(0.4) \text{ is the number of women that are added to the team so that four tenths of the team is female, which for this team would be the addition of 40 women.}
\]
6. [11 points] The Dow Jones Industrial Average (commonly referred to as “The Dow”) monitors the value of the 30 most popularly traded stocks in the country and generally indicates the status of the overall economy\(^1\). The following table shows the approximate closing value \(D(x)\), measured in thousands, of the Dow \(x\) months after June 6, 2008\(^2\).

<table>
<thead>
<tr>
<th>months after 6/6/08, (x)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dow Jones closing value (in thousands), (D(x))</td>
<td>12.2</td>
<td>11.3</td>
<td>11.3</td>
<td>11.2</td>
<td>10.3</td>
<td>8.9</td>
<td>8.6</td>
</tr>
</tbody>
</table>

\(a\). [3 points] Using your calculator, find the equation for \(L(x)\), the least-squares linear regression that approximates the value of the Dow \(x\) months after June 6, 2008.

\[L(x) = -0.59285x + 12.32143\]

\(b\). [2 points] Give a real-world interpretation for the slope of the regression line you found in part (a).

\[Solution:\] Each additional month after June 6, 2008, the Dow loses approximately 0.59285 thousand (or 592.85) points in value.

\(c\). [3 points] Using your calculator, find the equation for \(E(x)\), the exponential regression curve that approximates the value of the Dow \(x\) months after June 6, 2008.

\[E(x) = 12.44868(0.94381)^x\]

\(d\). [3 points] On March 6, 2009, the Dow closed at approximately 6,600. Throughout March and into April, it appeared that the market started growing again after several months of consecutive losses. On April 6, 2009, the Dow closed at approximately 8,000. If the Dow continues to grow, will either of the regression curves you found in parts (a) and (c) provide useful approximations for future closing values? Justify your answer in one complete sentence.

\[Solution:\] Neither regression curve would provide a useful approximation because both are decreasing functions for all values of \(x\) and would not accurately model long-term increases in the Dow.

\(^1\)http://wordnet.princeton.edu/perl/webwn
\(^2\)http://dowjonesclose.com
7. [13 points] Brian places a sensor on a spoke of the front wheel of his bicycle that measures his speed and the distance he travels. At its lowest level, the sensor is 2.5 inches above ground level. At its highest level, the sensor is 25.5 inches above ground level.

a. [4 points] At \( t = 0 \), the sensor is 2.5 inches above the ground. Brian is riding his bicycle such that the wheel rotates 1.5 times per second. In the space provided below, draw a graph of \( h(t) \), the height, in inches, of the sensor above the ground at \( t \) seconds. Draw your graph over the domain \( 0 \leq t \leq 2 \), and be sure to label your axes and important features.

\[
\begin{align*}
0 & \quad 0.5 & \quad 1 & \quad 1.5 & \quad 2 \\
0 & \quad 2.5 & \quad 14 & \quad 25.5
\end{align*}
\]
\[t, \text{time (seconds)} \]
\[h, \text{height (inches)} \]

\[
\begin{align*}
\text{Solution:}
\end{align*}
\]

b. [4 points] Using your graph above, find a formula for \( h(t) \).

\[
\text{Solution:} \quad h(t) = -11.5 \cos(3\pi t) + 14
\]

c. [5 points] In the wheel’s first rotation, find two times at which the sensor is 20 inches above the ground. Use algebra to solve for the exact answers, and show all work.

\[
\text{Solution:} \quad \text{We want to solve } h(t) = -11.5 \cos(3\pi t) + 14 = 20. \text{ We have}
\]

\[
\begin{align*}
-11.5 \cos(3\pi t) + 14 & = 20 \\
-11.5 \cos(3\pi t) & = 6 \\
\cos(3\pi t) & = \frac{-6}{11.5} \\
3\pi t & = \cos^{-1} \left( \frac{-6}{11.5} \right) \\
t & = \frac{1}{3\pi} \cos^{-1} \left( \frac{-6}{11.5} \right)
\end{align*}
\]

The second solution occurs at \( t = \frac{2}{3} - \frac{1}{3\pi} \cos^{-1} \left( \frac{-6}{11.5} \right) \).
8. [9 points] A cold item’s temperature will eventually approach the temperature of its warmer surroundings. Suppose a cold bottle of water has a temperature of 36°F when it is removed from a refrigerator at 8:00 a.m. It is then placed on the counter in a kitchen that is 68°F and left there. The temperature of the water, in degrees Fahrenheit, \( t \) hours after 8:00 a.m. is given by \( W(t) \), where

\[
W(t) = 68 - 32e^{-0.8t}.
\]

a. [3 points] The function \( W(t) \) is a transformation of the function \( f(t) = e^t \). Listed below are the correct transformations that must occur to the graph of \( f(t) \) in order to obtain the graph of \( W(t) \). In the blank space next to each step, specify which step should occur first by writing ”1”, which one should be second by writing ”2”, and which should be third by writing ”3”. You do not need to show your work for this problem.

1. vertical shift up 68 units
2. vertical stretch by a factor of 32 accompanied by a vertical reflection over the x-axis
3. horizontal stretch by a factor of \( \frac{1}{0.8} \) accompanied by a horizontal reflection over the y-axis.

b. [6 points] Using algebra, analytically solve for when the temperature of the water is 60°F. For your final answer, give the time of day at which this occurs to the nearest minute. Be sure to show all of your work.

**Solution:** We want to solve \( W(t) = 68 - 32e^{-0.8t} = 60. \) We have

\[
\begin{align*}
68 - 32e^{-0.8t} &= 60 \\
-32e^{-0.8t} &= -8 \\
e^{-0.8t} &= \frac{1}{4} \\
-0.8t &= \ln\left(\frac{1}{4}\right) \\
t &= -\frac{1}{0.8} \ln\left(\frac{1}{4}\right) \\
&= -1.25 \ln(0.25)
\end{align*}
\]

This is approximately 1.7328 hours after 8:00 a.m., when the water was placed on the counter. So the water is 60°F at 9:43-9:44 a.m. (depending on how you round your answer).
9. [12 points] The function \( f(x) \) goes through the points \((1,9)\) and \((3,1)\).

a. [4 points] Find a formula for \( f(x) \) if it is a linear function.

**Solution:** We want to find a formula of the form \( f(x) = mx + b \). The slope is found by

\[
m = \frac{1 - 9}{3 - 1} = -4.
\]

Then we have \( f(x) = -4x + b \), and using \( f(1) = 9 \), we find that \( b = 13 \) so that

\[
f(x) = -4x + 13.
\]

b. [4 points] Find a formula for \( f(x) \) if it is an exponential function.

**Solution:** We want to find a formula of the form \( f(x) = ab^x \). First we solve for the growth factor, \( b \). We have

\[
\frac{f(3)}{f(1)} = \frac{ab^3}{ab} = b^2,
\]

but we also know \( \frac{f(3)}{f(1)} = \frac{1}{9} \). So we have \( b^2 = \frac{1}{9} \), which gives \( b = \frac{1}{3} \). We use \( f(1) = 9 \) to solve for the initial value, \( a \). We have

\[
f(1) = a \left( \frac{1}{3} \right)^1 = 9,
\]

so \( a = 27 \). That gives us

\[
f(x) = 27 \left( \frac{1}{3} \right)^x.
\]

c. [4 points] Find a formula for \( f(x) \) if it is a power function.

**Solution:** We want to find a formula of the form \( f(x) = ax^p \). First we solve for \( p \) by using

\[
\frac{f(3)}{f(1)} = \frac{a3^p}{a1^p} = \left( \frac{3}{1} \right)^p = (3)^p.
\]

Since we know \( \frac{f(3)}{f(1)} = \frac{1}{9} \), we have \( 3^p = \frac{1}{9} \), so \( p = -2 \). Now we use \( f(1) = 9 \) to solve for \( a \). We have \( f(1) = a1^{-2} = 9 \), so \( a = 9 \). This gives us

\[
f(x) = 9x^{-2}.
\]
10. [10 points] In the year 1750, 2375 people founded the town of Arborville. The town’s population \( t \) years after it was founded is shown in the figure below, where \( P(t) \) is the number of people in the town.

![Graph of population over time](image)

**a.** [6 points] The function \( P(t) \) shown above is a cubic polynomial. Find a formula for \( P(t) \).

\[
\text{Solution: } P(t) \text{ is another polynomial, call it } f(t) \text{ shifted upward 3000 units. First we can find a formula for } f(t), \text{ which would have a zero at } t = 5 \text{ of even multiplicity and another zero at } t = 25 \text{ (no multiplicity). This would give us } f(t) = k(t - 5)^2(t - 25), \text{ where } k \text{ is a stretch factor. Since } P(0) = 2375, \text{ we know that } f(0) = -625. \text{ So we can solve for } k \text{ by having } f(0) = k(0 - 5)^2(0 - 25) = -625, \text{ which gives us } k = 1. \text{ Lastly, we remember that we obtain } P(t) \text{ by shifting } f(t) \text{ upward by 3000 units so that } P(t) = (t - 5)^2(t - 25) + 3000 = t^3 - 35t^2 + 275t + 2375.\]

**b.** [4 points] During the first 30 years of the town’s existence, what was the minimum population? In what year was the population this minimum value? State your answers in a complete sentence.

\[
\text{Solution: } \text{We can use the ability to calculate a minimum on the graphing calculator to find the minimum population. The minimum population is approximately 1814.8, or 1815 people, which occurred approximately 18.3 years after 1750, or in 1768.}\]