MATH 105 — FINAL EXAM

DEPARTMENT OF MATHEMATICS
University of Michigan

December 13, 2002, 8 am - 10 am

NAME: ___________________________  ID NUMBER: _______________________

SIGNATURE: ______________________

INSTRUCTOR: ______________________  SECTION NO: _______________________

General Instructions: Do not open this exam until you are told to begin. The exam consists of 13 questions on 8 pages (including this cover). The exam has a 100 point scale. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam. Show an appropriate amount of work for each exercise so that graders can see not only the answer but also how you obtained it. Unless explicitly stated, no credit will be given for answers that do not show how they are derived. If you use graphs or tables to obtain an answer, be certain to provide an explanation (and a sketch of the graph, if that is the method) to make it clear how you arrived at your solution. Use units where appropriate. You may use your calculator, but no other outside materials.

<table>
<thead>
<tr>
<th>PROBLEM</th>
<th>POINTS</th>
<th>SCORE</th>
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<td>1</td>
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<td>2, 3 &amp; 4</td>
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<td>5, 6 &amp; 7</td>
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<td>8 &amp; 9</td>
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<td>10 &amp; 11</td>
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<td>13</td>
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<td>TOTAL</td>
<td>100</td>
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</table>
1. (a) (4 points) Write a possible formula for this line.

\[ y = \quad \]

b) (4 points) Write a possible formula for this parabola.

\[ y = \quad \]

c) (6 points) Write a possible formula for this polynomial.

\[ y = \quad \]
2. (a) (4 points) Write a possible formula for this power function.

\[ y = \frac{1}{x} \]

b) (2 points) The domain of this function is \( \mathbb{R} \setminus \{0\} \).

3. (a) (4 points) Write a possible formula for this trigonometric function.

\[ y = \sin x \]

b) (2 points) The range of this function is \([-1, 1]\).

4. (2 points) In general, for any function, \( f(x) \), what points (if any) on the graph of \( f \) do not change when the graph is stretched vertically?
5. (5 points) A person breathes in and out every 3 seconds. The volume of air in the person’s lungs varies between a minimum of 2 liters and a maximum of 4 liters. Circle the best formula for the volume of air in the person’s lungs as a function of time $t$ in seconds.

$y = 2 + 2 \sin \left( \frac{\pi}{3} t \right)$  
$y = 2 + 2 \sin \left( \frac{2\pi}{3} t \right)$  
$y = 3 + \sin \left( \frac{\pi}{3} t \right)$  
$y = 3 + \sin \left( \frac{2\pi}{3} t \right)$

6. By considering the following three functions as $x$ takes on larger and larger positive values, fill in the blanks below.

\[
\frac{x^4 + 1}{x - 2}, \quad 7x^4 + 3x^3, \quad 1.005^x
\]

a) (3 points) The function which eventually grows the fastest is ____________________.

b) (3 points) The function which eventually grows the slowest is ____________________.

7. This question concerns the functions in the list below. Write the appropriate letter(s).

A) $y = \frac{1}{2} + \sin(x^3)$  
B) $y = 3 \sin \left( \frac{x}{5} - \pi \right)$  
C) $y = 4^{3x}$  
D) $y = 3 + \frac{2}{x}$  
E) $y = \frac{7x^3 + 3x}{(x + 1)^2}$  
F) $y = \frac{\sqrt{x}}{x + 1}$  
G) $y = \frac{\sin x}{\cos x}$  
H) $y = \frac{1}{e^x}$  
I) $y = e^{x^2}$

a) (2 points) Which of these functions are rational functions?

b) (2 points) Which of these functions are exponential functions?

c) (2 points) Which of these functions are trigonometric functions?
8. Consider the function \( f(x) = \sqrt[3]{\frac{x}{2}} + 1 \).

a) (3 points) Find the formula for \( f^{-1}(x) \).

b) (3 points) Demonstrate directly that the functions \( f \) and \( f^{-1} \) are inverses by showing that 
\[ f^{-1}(f(x)) = x. \]

9. The graph of a function \( Q \) is given below.

a) (4 points) Identify one specific characteristic of this graph which guarantees that \( Q \) is not a trigonometric function. \textbf{Briefly explain in a full sentence.}

b) (4 points) Identify one specific characteristic of this graph which guarantees that \( Q \) is not an exponential function. \textbf{Briefly explain in a full sentence.}
10. Let \( g(x) = 3x + 5 \).

a) (2 points) Determine the equation for \( h(x) = g(x) - 2 \).

b) (2 points) Shifting the graph of \( g \) to the right 2 units is equivalent to shifting the graph of \( g \) vertically. Would this vertical shift be up or down? How many units?

11. Consider the functions \( f, g \) and \( h \) described by the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
<th>( h(x) )</th>
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<tbody>
<tr>
<td>-2</td>
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a) (2 points) Find a possible formula for \( f(x) \) in terms of \( x \).

b) (2 points) Find a possible formula for \( g(x) \) in terms of \( x \).

c) (2 points) Find a formula for \( h(x) \) in terms of \( f(x) \) and \( g(x) \).
12. The total worldwide wind energy generating capacity, $W$, was 1930 megawatts in 1990 and 18,100 megawatts in 2000.

a) (5 points) Write a formula for $W = G(t)$ as function of $t$, the number of years since 1990, assuming that $G$ has a constant average rate of change.

b) (5 points) Write a formula for $W = H(t)$ assuming that $H$ has a constant percent rate of change.

c) (5 points) Sketch a graph of these two functions on the axes below. Use scales which are appropriate to the situation being modelled and which also demonstrate what the models predict for the future. Be sure to label any important points.
13. The volume of pollutants (in millions of cubic feet) in a certain reservoir is given by

\[ P(t) = 361 + 8t, \]

where \( t \) is in years. The total volume of the reservoir (which includes both pollutants and water and also in millions of cubic feet) is gradually increasing and is given by

\[ R(t) = 12000 + 12t. \]

Let \( C(t) \) be the fraction of the reservoir’s total volume that consists of pollutants.

a) (2 points) Write an expression for \( C(t) = \frac{P(t)}{R(t)} \) in terms of \( t \).

b) (3 points) In year \( t = 0 \), what percentage of the reservoir’s total volume consists of pollutants?

c) (5 points) If these trends were to continue for many many years, about what percentage of the reservoir’s total volume would eventually consist of pollutants? **Briefly explain your reasoning in full sentences.**

d) (3 points) What is the value of \( C^{-1}(1/2) \)?

e) (3 points) Interpret \( C^{-1}(1/2) \) in terms of percentages and years.