Finding a Formula for a Polynomial Function: Roots and Multiplicities

A polynomial function is any function of the form:

\[ y = c_0 + c_1 \cdot x + c_2 \cdot x^2 + \ldots + c_n \cdot x^n \]

where the powers of \( x \) must be positive integers. The largest power of \( x \) in the polynomial is called the **degree** of the polynomial. Most graphing calculators can only fit a very limited number of polynomial functions to data. (For example, the TI-83 can only fit polynomials up to degree 4).

Luckily, it is often possible to use the graph to find an equation for a polynomial function. The key is that you have to be able to see all of the places where the graph cuts the \( x \)-axis, and you have to be able to clearly see the shape that the polynomial makes as it cuts through (see Figure 1).

The basic procedure is to:

1. Locate the “\( x \)-intercepts” or zeros of the polynomial function.
2. Determine the multiplicity of each zero.
3. Write down the “factored form” of the polynomial.
4. Use a point on the graph of the polynomial to determine the constant of proportionality, \( k \).
The example that follows illustrates the steps in this procedure.

Example: Finding the Equation for a Polynomial Function by Hand

Step 1: Locate the zeros

Inspection of Figure 2 shows that the zeros of this polynomial are located at \(x = -3\), \(x = -1\) and \(x = 2\).

Step 2: Determine the Multiplicity of Each Zeros

The multiplicity of the zero is determined by the appearance of graph near the zero (See Figure 3).

- If the graph looks as though it just cuts cleanly through the \(x\)-axis, then the zero has multiplicity one (see Figure 3(a)).
- If the graph looks like a quadratic and just touches the \(x\)-axis without cutting through, then the zero has multiplicity two (see Figure 3(b)).
- If the graph looks like a cubic and has an inflection point as it cuts through the \(x\)-axis, then the zero has multiplicity three (see Figure 3(c)).

For the function graphed in Figure 2, the multiplicity of the zeros are:
Step 3: Determine the Factored Form of the Polynomial

The factored form of the polynomial is an equation of the form:

\[ y = k \cdot (x - z_1)^{m_1} \cdot (x - z_2)^{m_2} \cdots (x - z_n)^{m_n}, \]

where \( k \) is called the “constant of proportionality, \( z_1, z_2, \ldots, z_n \) are the zeros of the polynomial function, and \( m_1, m_2, \ldots, m_n \) are the multiplicities of the zeros. The factored form of the polynomial from Figure 2 is:

\[ y = k \cdot (x + 3) \cdot (x + 1)^2 \cdot (x - 2)^3. \]

Step 4: Determine the Constant of Proportionality, \( k \)

The idea here is to locate the \( x \) and \( y \) coordinate of a point that is on the graph of the polynomial function, but which is not one of the zeros of the polynomial function. The \( x \) and \( y \) are substituted into the factored form, allowing \( k \) to be found.

From Figure 2, the point \((0, -2)\) lies on the graph of the function. Substituting this into the factored form gives: \( k = 1/12 \). Therefore, the equation for the polynomial function whose graph is show in Figure 2 is:

\[ y = \frac{1}{12} \cdot (x + 3) \cdot (x + 1)^2 \cdot (x - 2)^3. \]

Example

Find the formula of the polynomial function shown in Figure 4 (see over).

Solution

The zeros of the polynomial from Figure 4 are located at:

\[ x = -3, \text{ and } x = +2. \]

From the appearance of the graph in Figure 4 as the polynomial cuts through the \( x \)-axis, the multiplicities of the zeros are as follows:

<table>
<thead>
<tr>
<th>Zero located at ...</th>
<th>Multiplicity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = -3 )</td>
<td>1</td>
</tr>
<tr>
<td>( x = -1 )</td>
<td>2</td>
</tr>
<tr>
<td>( x = 2 )</td>
<td>3</td>
</tr>
</tbody>
</table>
Zero | Multiplicity
--- | ---
$x = −3$ | 1
$x = +2$ | 2

So, the factored form of the polynomial shown in Figure 4 is:

$$y = k \cdot (x - (-3))^1 \cdot (x - 2)^2.$$  

Re-writing this in a more conventional format gives the formula:

$$y = k \cdot (x + 3) \cdot (x - 2)^2.$$  

To determine the numerical value of $k$ we can use the fact that Figure 4 shows that the point $(0, 1)$ lies on the graph of the polynomial function. To determine $k$ we will substitute $x = 0$ and $y = 1$ into the factored form and solve for $k$. Doing this:

$$1 = k \cdot (0 + 3) \cdot (0 - 2)^2$$  

$$1 = k \cdot 12$$  

$$k = \frac{1}{12}.$$  

With the value of $k$ worked out, you can write down the complete formula for the polynomial function shown in Figure 4.

$$y = \frac{1}{12} \cdot (x + 3) \cdot (x - 2)^2.$$  

**Example**

Find the formula for the polynomial function shown in Figure 5 (see next page).

From Figure 5, the zeros of the polynomial function are:

$$x = −3 \text{ and } x = +1.$$  

The multiplicities of these zeros are 1 (for $x = −3$) and 3 (for $x = +1$).
The factored form of the formula for the polynomial in Figure 5 is then:

\[ y = k \cdot (x - (-3))^3 \cdot (x - 1)^3. \]

Writing this formula in slightly more conventional notation gives:

\[ y = k \cdot (x + 3) \cdot (x - 1)^3. \]

To determine the numerical value of the constant \( k \), you can use the fact that the point \((0, -1)\) lies on the graph.

To determine the numerical value of \( k \) you can plug \( x = 0 \) and \( y = -1 \) into the factored form, evaluate all of the quantities that you can, and then solve for \( k \). Doing this:

\[
-1 = k \cdot (0 + 3) \cdot (0 - 1)^3
\]
\[
-1 = k \cdot (-3)
\]
\[
k = \frac{1}{3}.
\]

Substituting this numerical value for the \( k \) in the factored form gives the formula for the polynomial function in Figure 5:

\[ y = \frac{1}{3} \cdot (x + 3) \cdot (x - 1)^3. \]