Math 115 - Team Homework Assignment #4, Fall 2019

- **Due Date:** October 24 or 25, 2019 (Your instructor will tell you the exact date and time.)
- It is important that you try these problems **before your first meeting** with your team. Remember, the first meeting with your team is for the members of the team to discuss the arguments used by each member of the team to find their solutions. It is also a time to address any questions that each member of the team may have about the assignment.
- The team homework roles are
  - **Manager:** organizes and runs the meetings.
  - **Reporter:** writes a record of the meeting, including the roles of every member, how often the team met, how the assignment went, and any challenges and successes the team encountered. This reporter sheet is attached as the top sheet of your final draft.
  - **Scribe:** writes up a single final draft of the homework assignment to turn in.
  - **Clarifier:** makes sure everyone is on the same page and understands the problem’s statements and solutions.

Assign roles to each team member before beginning the team homework and rotate these roles between team members for every assignment. If your team has only three members, combine the roles of manager and clarifier into one; if it has five members, there should be two clarifiers.

- Please refer to [https://instruct.math.lsa.umich.edu/support/teamhomework/](https://instruct.math.lsa.umich.edu/support/teamhomework/) for full details on writing team homeworks.
- Highlighted text will be used to add commentary to the problems. In most cases this can be safely skipped, but you are encouraged to read it to gain context for both the problem and, hopefully, the course as a whole.

Please complete this checklist before submitting your team homework.

☐ Our solution includes a **restatement of the problem** in our own words, the way we understood it.  
  It would be clear to someone who has not seen the question what the set up is and what we will do in our solution. In particular, all functions and variables have been defined, all graphs, formulas, and tables have been reproduced.

☐ Our solution is written in **full sentences**.  
  In particular, all our computations are part of sentences, and every step is clear.

☐ Our solution includes **all the steps** we took to arrive at the answer.  
  In particular, all the calculations, graphs, tables we used are included.

☐ Every step of our solution is **thoroughly explained** and **justified**.  
  Another Math 115 student, who did not understand how to solve this question before, would understand it after reading our solution.
1. Jumping around...

Let \( w(x) \) be a continuous function defined on the interval \(-7 \leq x \leq 7\). You are given the graph of \( w'(x) \), the derivative of \( w(x) \), below.

(a) For each of the following, give all values (or intervals) of \( x \) in \((-7, 7)\) for which the statement is valid at that value (or on that interval).

i. \( w(x) \) is increasing.

ii. \( w(x) \) is not differentiable.

iii. \( w(x) \) is concave up.

iv. \( w''(x) \) does not exist.

v. \( w''(x) \) is decreasing.

(b) Draw a careful, well-labeled sketch of \( w''(x) \). Pay close attention to the following:

- where \( w''(x) \) is defined
- the \( y \)-values on your graph of \( y = w''(x) \)
- where \( w''(x) \) is increasing, decreasing, or constant.

(c) Assume that \( g(x) \) is a differentiable function defined on the interval \(-7 \leq x \leq 7\). Some values of \( g(x) \) and \( g'(x) \) are given below.

\[
\begin{array}{c|cccccc}
    x & -7 & -6 & 1 & 2 & 4 & 7 \\
    g(x) & 0 & 0.25 & -1 & 1 & 2 & 0.4 \\
    g'(x) & -1.5 & 2 & 4 & -3 & 5 & -7 \\
\end{array}
\]

Assume that the graph of \( w(x) \) passes through the point \((4, 8)\). Compute each of the following, or explain why the quantity does not exist or why you do not have enough information to compute it.

i. \( H'(-6) \) where \( H(x) = 7 \sin(\pi g(x)) \cdot w'(x) \)

ii. \( k'(4) \) where \( k(x) = (-g(x) + 3)^3 \cdot w(x) \)

iii. \( U'(4) \) where \( U(x) = \frac{g(x - 3)}{w'(x)} \)

iv. \( \frac{d}{dx} \left( g(x + w'(x)) \right) \bigg|_{x=2} \)
Consider the semester so far: we built a foundation by studying limits, then defined the derivative using a limit, and are now exploring the many ways derivatives are useful in practical applications. The historical development of calculus, on the other hand, proceeded in the opposite direction.

**Isaac Newton** (1642–1727) and **Gottfried Leibniz** (1646–1716) are both credited with developing the idea of the derivative as a means to solve practical problems in physics. Newton’s work was very geometric, and studied derivatives through tangent lines. Never very careful about notation, Newton used whatever was most convenient at the time, leading to both the “prime” notation ($f'$) we use, as well the “dot” notation ($\dot{y}$) used in physics. In contrast, Leibniz approached the derivative analytically in terms of difference quotients. This thinking led to his very careful and consistent use of the other notation we’ve seen, $\frac{dy}{dx}$.

Newton and Leibniz’s ideas, while extremely useful, were also problematic, leading to many logical inconsistencies. It took over 100 more years for the idea of a limit to be developed, finally placing calculus onto a rigorous foundation.

Consider the function below, defined for $0 < x < \pi$, where $a$ and $b$ are constants.

$$g(x) = \begin{cases} 
\frac{b x^2}{\pi^2} + 6 & 0 < x < \frac{\pi}{2} \\
 a \left( \frac{2x}{\pi} \right)^a + \left( \frac{5b}{4} \right) \sin \left( \frac{x}{\pi} \right) & \frac{\pi}{2} \leq x < \pi 
\end{cases}$$

(a) Give an example of values that could be plugged into $a$ and $b$ in order to make $g(x)$ a continuous function.

(b) Is the function differentiable when $a = 2$ and $b = 8$? Explain your answer. You may want to check your answer using a graphing utility such as Desmos or your calculator.

(c) Find all values of $a$ and $b$ that make $g(x)$ a differentiable function. You must carefully justify your answer. *Hint: There are only two pairs of values $(a, b)$ that will work.*

(d) Using a graphing utility like Desmos, sketch a graph of $g(x)$ using each of possible values for the constants $a$ and $b$ you found in the previous part.
3. Don’t look a gift horse in the mouth

As a birthday present, your friend (a Math 115 student) gave you a graph consisting of a function, its derivative, and its second derivative. Unfortunately, your friend forgot to label which graph is which. It would be rude to ask your friend after they worked so hard to put this gift together, so you decide to add your own labels, “I”, “II”, and “III”, and figure it out for yourself.

Graph I is shown as the dotted, red curve. Graph II is shown as the solid, green curve. Graph III is shown as the dashed, blue curve.

(a) One of the three above graphs is the function $f(x)$, one is the function $f'(x)$, and the other is the function $f''(x)$. Which graph corresponds to which function? Explain your answer.

(b) You open your birthday card and find the following mathematical problems inside. Your answers may consist of one or more intervals. Exact answers are not necessary – estimate any relevant endpoints as best you can with the information given.

i. Where is $f''(x)$ concave up?
ii. Assume $q(x)$ is a function satisfying $q'(x) = e^{-2f(x)}$. Where is $q(x)$ concave up?
iii. Let $c(x) = 5(f'(x))^2$. Where is $c(x)$ increasing?