1. **Football Espionage**

The Cornhusker Intelligence Agency (CIA) at the University of Nebraska is conducting an electronic surveillance operation on the Michigan football program during their training season, and is attempting to determine Michigan’s running capabilities. The CIA successfully hacked into the running watch that a Michigan football player was wearing, but could only access data about direction and speed. From this graph, the CIA would like to learn more about the player’s position.

Assume that the Michigan football player was running back and forth along the sideline, which stretches between the north goal line and the south goal line 100 yards away. Let $v(t)$ be the runner’s velocity in yards per second $t$ seconds after he started running, as measured by the running watch. Positive velocity indicates movement towards the north goal line.

On the interval $[30, 50]$, the function $v(t)$ is given by the formula

$$v(t) = \frac{3}{100} t^2 - \frac{12}{5} t + 45.$$

(a) Let $p(t)$ be the player’s distance from the south goal line, measured in yards, $t$ seconds after he started running. Assume that the player begins at $p(0) = 10$ yards from the south goal line. Sketch a graph of the player’s position over the interval $0 \leq t \leq 60$. Pay careful attention to where your graph is increasing or decreasing, where it is concave up or concave down, and to the exact value of the runner’s position at the following times: $t = 0, 10, 20, 30, 40, 50$ and $60$.

**Solution:** We’ll find the exact values of $p(t)$ by using the Fundamental Theorem of Calculus (FTC), which says that

$$\int_a^b v(t) \, dt = \int_a^b p'(t) \, dt = p(b) - p(a)$$

since $v(t) = p'(t)$, that is, $p$ is an antiderivative of $v$. 
Using the formulas for the areas of a triangle and rectangle, we see that
\[ \int_0^{10} v(s) \, dt = \frac{1}{2} \cdot 10 \cdot 1 = 5 \]
\[ \int_{10}^{20} v(t) \, dt = 10 \cdot 1 + \frac{1}{2} \cdot 10 \cdot 4 = 30 \]
\[ \int_{20}^{30} v(t) \, dt = 10 \cdot 5 = 50. \]

So, using the FTC on \([0, 10]\), we know that
\[ 5 = \int_0^{10} v(t) \, dt = p(10) - p(0) = p(10) - 10, \]
so that \(p(10) = 15\). Similarly we can then use the FTC on \([10, 20]\) and \([20, 30]\) to find that \(p(20) = 45\) and \(p(30) = 95\).

To find the integral from \([30, 40]\) we will use the formula provided, properties of integrals, and our knowledge of antiderivatives of power functions:
\[
\int_{30}^{40} v(t) \, dt = \frac{3}{100} \int_{30}^{40} \frac{t^2}{3} - \frac{12}{5} t + 45 \, dt
\]
\[= \frac{3}{100} \left( \frac{t^3}{3} \bigg|_{30}^{40} - \frac{12}{5} \cdot \frac{t^2}{2} \bigg|_{30}^{40} + 45t \bigg|_{30}^{40} \right) \]
\[= \frac{1}{100} (40^3 - 30^3) - \frac{6}{5} (40^2 - 30^2) + 45(40 - 30) \]
\[= \frac{37,000}{100} - \frac{6 \cdot 700}{5} + 450 \]
\[= -20. \]

By symmetry of the parabola we also know we must have
\[ \int_{30}^{40} v(t) \, dt = \int_{40}^{50} v(t) \, dt = -20. \]

Now, using the FTC on \([30, 40]\), we see that
\[ -20 = \int_{30}^{40} v(t) \, dt = p(40) - p(30) = p(40) - 95, \]
so \(p(40) = 75\). Similarly, the FTC on \([40, 50]\) gives us \(p(50) = 55\).

Lastly, the integral from 50 to 60 is just a rectangle, so
\[ 10 \cdot 2 = \int_{50}^{60} v(t) \, dt = p(60) - 55, \]
so \(p(60) = 75\). That is, we have found the following table of values for \(p(t)\):

<table>
<thead>
<tr>
<th>(t)</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p(t))</td>
<td>10</td>
<td>15</td>
<td>45</td>
<td>95</td>
<td>75</td>
<td>55</td>
<td>75</td>
</tr>
</tbody>
</table>
Since $v = p'$ is positive on $(0, 30)$, our sketch of $p$ should be increasing on this interval, and again on $(50, 60)$, and since $v = p'$ is negative on $(30, 50)$, our graph of $p$ should be decreasing there.

Now we consider concavity. Since $v = p'$ is increasing on $[0, 20]$, $p$ is concave up on $[0, 20]$. The velocity $v = p'$ is constant on $[20, 30]$, thus $p$ is a straight line with slope 5. The velocity $v = p'$ is decreasing on $[30, 40]$ and increasing from $[40, 50]$ so $p$ is concave down on $[30, 40]$ and concave up on $[40, 50]$. Again $p$ has no concavity on $[50, 60]$, and instead is linear with slope 2.

Thus we obtain the sketch $p(t)$ shown below.

(b) When did the player first reach a position of 50 yards from the south goal line? Does he ever reach this position again during the first minute?

**Solution:** From the table above we see the player runs from the 10 to the 45 yard line, reaching it at $t = 20$ seconds. Since the player is running at constant velocity of 5 yards/sec during the time interval $[20, 30]$, in the next second he will run an additional 5 yards. Thus, at $t = 21$ he will be at the 50 yard line.

He will not return to the 50 yard line again. The player runs back towards the south goal line when he has negative velocity, but this only occurs between $t = 30$ and $t = 50$, and at $t = 50$ we know he only made it back to the 55 yard line, at which point he starts running back towards the north goal line again for the remainder of the first minute.

Note that our sketch shows this as well: the only time the graph of $p(t)$ has a $y$-value of 50 seems to be at about $t = 21$. 
(c) One minute after he started running, how far from his starting position is the player located?

Solution: At $t = 60$, the player is 75 yards north of the south goal line, but he started 10 yards north of the south goal line. That is, the player is $p(60) - p(0) = 75 - 10 = 65$ yards away from his starting position.

(d) What was the total number of yards that the player ran during this minute?

Solution: The total number of the yards the player runs is given by

$$\int_{0}^{60} |v(t)| dt.$$  

That is, we should look at the areas we computed in part (a) and ignore any negative signs. Thus, the total distance the player traveled was $5 + 30 + 50 + 20 + 20 + 20 = 145$ yards.
2. Save the aliens!
A pig slop manufacturing giant has (allegedly) been dumping thousands of gallons of extremely
dangerous radioactive waste into a local alien wildlife park in Rachel, NV (near Area 51). The city
declared a public health emergency and immediately started intense cleanup efforts over the next
four days. Unfortunately, even as they were cleaning up, new waste was still spewing out from an
illegal hidden pig slop pipeline.

- Let \( W(t) \) be the rate, in thousands of gallons per hour, at which new radioactive waste was
  being dumped into the site at time \( t \) hours since the cleanup began.
- Let \( C(t) \) be the rate, in thousands of gallons per hour, at which radioactive waste was being
  removed from the site by cleanup crews at time \( t \) hours after the cleanup began.

Time \( t = 0 \) is exactly midnight. A graph of \( W(t) \) is shown below. At the time when Rachel city
officials discovered the illegal dumping site, there were already 40 thousand gallons of waste present.

You do not have a graph of \( C(t) \). Instead, you have the following pieces of information:

- Because of the Nevada heat, cleanup crews are most active in the AM shift, which runs from
  midnight to noon. The PM shift runs from noon to midnight.
- There are 30 workers on-site during each of the first two AM shifts, and 60 workers on site
  during the last two AM shifts. Every AM shift worker removes waste at a constant rate of 0.05
  thousands of gallons per hour.
- There are 5 workers on-site during the first PM shift. Each subsequent day adds 5 more
  workers (with 20 total in the last PM shift). Every PM shift worker removes waste at a
  constant rate of 0.1 thousands of gallons per hour.

Answer each of the following.

(a) Sketch a graph of \( C(t) \) on the same axes as \( W(t) \).

**Solution:** The first AM shift occurs from \( t = 0 \) to \( t = 12 \). Since each of the 30 workers
can remove 0.05 thousands of gallons per hour, the entire crew can remove waste at a rate
of \( 30 \cdot 0.05 = 1.5 \) thousands of gallons per hour. That is, \( C(t) = 1.5 \) on \((0,12)\) as graphed
below. With the given information, we see that \( C(t) = 1.5 \) on \((24,36)\) as well. The other
two AM shifts double the cleanup rate, so \( C(t) = 3 \) on \((48,60)\) and \((72,84)\).

The value of \( C(t) \) can be found during the PM shifts similarly by multiplying the number
of workers present by the cleanup rate per worker. We get \( C(t) = 0.5 \) on \((12,24)\), \( C(t) = 1 \)
on \((36,48)\), \( C(t) = 1.5 \) on \((60,72)\), and \( C(t) = 2 \) on \((84,96)\).
The graph of $C(t)$ is shown below, with the AM shifts shown in red and the PM shifts in blue. Note that it doesn’t really matter whether you considered $t = 12$ as part of the AM or PM shift, but make sure you haven’t defined $C(12)$ to be both 1.5 and 0.5. That is, you should only have one closed dot, not two, at $t = 12$, and similarly for $t = 24, 36$, etc.

(b) Give a practical interpretation of the integral $\int_{24}^{48} C(t) \, dt$ in the context of the problem.

**Solution:** Since $C(t)$ is the rate at which waste is removed from the site, by the FTC this integral is the total amount of waste removed between $t = 24$ and $t = 48$. Since the value of this integral is $1.5 \cdot 12 + 1 \cdot 12 = 30$ thousand gallons, one practical interpretation is

> During the second day of the cleanup effort, 30 thousand gallons of radioactive waste were removed from the site.

(c) Over the course of the first two days of the cleanup effort, what was the average rate at which new waste was being dumped into the site? Over the same interval of time, what was the average rate at which waste was being removed by the workers?

**Solution:** The average rate at which waste is dumped into the site during the first two days is given by

$$\frac{1}{48 - 0} \int_{0}^{48} W(t) \, dt,$$

which we can compute geometrically. The area between $W(t)$ and the $t$-axis on $(0, 24)$ is a trapezoid of area $\frac{2+4}{2}(24) = 72$ (or use a triangle and rectangle). The area on $(24, 30)$ is a trapezoid of area $\frac{1+3}{2}(6) = 18$; and the corresponding area on $(30, 48)$ is a rectangle of area $(0.5)(18) = 9$. Summing these gives the value of our integral, 99. The average rate at which waste is dumped into the site over this period of time is thus $\frac{99}{15} = \frac{33}{5}$ thousands of gallons per hour.

Similarly, the average rate at which waste is removed during this time is

$$\frac{1}{48 - 0} \int_{0}^{48} C(t) \, dt.$$

To compute this integral, we add up the areas of the four rectangles we see on the interval $(0, 48)$ and get $(1.5)(12) + (0.5)(12) + (1.5)(12) + (1)(12) = 54$, so the average rate at which waste is cleaned up during this time is $\frac{54}{48} = \frac{9}{8}$ thousands of gallons per hour.
(d) At what time $t \geq 0$ was the amount of radioactive waste at the work site the greatest? How much waste was present at this time? Be sure to fully justify your answer.

**Solution:**
Let’s define the amount of radioactive waste at the site at time $t$ by the new function $R(t)$. Note that $R'(t)$, the derivative of $R(t)$, is given by $R'(t) = W(t) - C(t)$. To find the global maximum of $R(t)$ on $[0, 96]$, we should test all critical points of $R(t)$ and both endpoints. There are many, many critical points, however, so before doing all that work, let’s see if we can reason out our answer differently.

First, we see that $R'(t) = W(t) - C(t)$ is positive from $t = 0$ until $t = 30$, then negative until $t = 66$, then positive again for a short while until $t = 72$, then negative for the remainder of the interval. That means that $R(t)$, the total amount of waste at the site, is increasing on $[0, 30]$, decreasing on $[30, 66]$, increasing on $[66, 82]$, and decreasing again on $[72, 96]$. So the only possible candidates for the global maximum are at $t = 30$ and $t = 72$.

Now, by the FTC, we know that the change in the amount of waste at the site over the interval $[0, 30]$ is given by

$$
\int_{0}^{30} R'(t) \, dt = \int_{0}^{30} W(t) - C(t) \, dt,
$$

and similarly on other intervals. That is, it’s given by the area between $W(t)$ and $C(t)$ on our graph. So over $[0, 30]$, the amount of waste increased by some amount, equivalent to about 9 boxes. (We will compute this exactly below.) Then over $[30, 66]$, the amount of waste decreases, by about 7 or so boxes. Then over $[66, 72]$, the amount of waste increases again, but not even by one box. This means that the amount of waste at $t = 30$ must be significantly larger than at $t = 72$. Remember, over $[72, 96]$, the amount of waste decreases again, so the amount of waste must be largest at $t = 30$.

Now, let’s find how much waste was present at $t = 30$. Because the site started with 40 thousand gallons of waste present at the site (be careful, the initial amount can be easy to forget!), the total amount of waste present at $t = 30$ will be

$$
40 + \int_{0}^{30} R'(t) \, dt = 40 + \int_{0}^{30} W(t) - C(t) \, dt = 40 + \int_{0}^{30} W(t) \, dt - \int_{0}^{30} C(t) \, dt.
$$

We can now find the values of these two integrals geometrically by reusing some of our work from part (c): the first is equal to 72 + 18 = 90, while the second is equal to $(1.5)(12) + (0.5)(12) + (1.5)(6) = 33$. Thus, the total amount present at that time will be $40 + 90 - 33 = 97$ thousand gallons. Note that we could have also computed the area between the two curves directly.
(e) At time $t = 96$ hours, the workers found and managed to shut off the hidden pipeline. How much radioactive waste did the pig slop manufacturer dump in total?

**Solution:** The total amount dumped into the site over these four days is given by the integral

$$
\int_{0}^{96} W(t) \, dt,
$$

which we can find geometrically. (Note that we’ve already done about half the work in part (c).) You should find that the total amount of waste dumped over these four days is 135 thousand gallons.

(f) How much waste was left at the end of the 4 day cleanup effort?

**Solution:** Again using the FTC, we know that the total amount of waste present at the site at $t = 96$ is given by

$$
40 + \int_{0}^{96} W(t) \, dt - \int_{0}^{96} C(t) \, dt.
$$

We found the value of the first of these integrals in part (e); the second can be found similarly to be 168. Thus, the total amount of waste present at the end of the cleanup will be $40 + 135 - 168 = 7$ thousand gallons.
3. It actually is just a weather balloon.
You are a meteorologist working for the National Weather Service (NWS), collecting data from a weather balloon. An important piece of information is the Convective Available Potential Energy (CAPE) value, measured in units of Joules per kilogram (J/kg). A large CAPE value indicates a high risk of severe thunderstorms, tornadoes, and other catastrophic weather events.

- Let \( A(h) \) be the ambient temperature, in degrees Celsius (°C), measured by the weather balloon at a height of \( h \) km above the surface.
- Let \( P(h) \) be the internal temperature, in °C, that a cloud of surface air (called a parcel) would be if it rose to a height of \( h \) km. This is a function that meteorologists can model using other pieces of data collected by the weather balloon.

After launching your weather balloon earlier today, you obtained the following graph of the functions \( T = A(h) \) (solid and red) and \( T = P(h) \) (dashed and blue).

Meteorologists call the atmosphere unstable if air rising up from the surface can become warmer than the ambient air high up in the atmosphere, that is, if \( P(h) > A(h) \). The CAPE value is an integral taken over the heights \( h \) where \( P(h) > A(h) \). For this data, the CAPE value is therefore defined as follows:

\[
\text{CAPE} = 9800 \int_{3}^{11} \left( \frac{P(h)-A(h)}{A(h)+273} \right) \, dh \text{ J/kg.}
\]

The atmosphere is said to have strong instability if \( 2500 < \text{CAPE} \leq 4000 \) and extreme instability if \( \text{CAPE} > 4000 \).

(a) Estimate the CAPE value using left-hand and right-hand Riemann sums with 4 equal subdivisions. Does either of these estimates reach the threshold of strong instability? Do either reach the threshold of extreme instability?

**Solution:** As we will do in part (b), let’s define

\[
B(h) = \frac{P(h)-A(h)}{A(h)+273}.
\]
Since the interval \([3, 11]\) has width \(11 - 3 = 8\), to have four subdivisions we’ll need to use \(\Delta h = 2\). So to find the Riemann sums we need, we need to estimate \(B(h)\) at \(h = 3, 5, 7, 9,\) and \(11\). From the graph, we estimate that that \(A(3) = 10\) \(A(5) = -10\) \(A(7) = -20\) \(A(9) = -20\) \(A(11) = -20\)

\[
P(3) = 10 \quad P(5) = 5 \quad P(7) = 0 \quad P(9) = -10 \quad P(11) = -20,
\]

so that

\[
\begin{align*}
B(3) &= 0 \\
B(5) &= \frac{15}{263} \approx 0.057 \\
B(7) &= \frac{20}{253} \approx 0.079 \\
B(9) &= \frac{10}{253} \approx 0.0395 \\
B(11) &= 0.
\end{align*}
\]

Remembering both the constant of 9800 out front and to multiply each value by \(\Delta h\), we find that the left-hand Riemann sum on the interval \([3, 11]\) is therefore given by

\[
9800(2B(3) + 2B(5) + 2B(7) + 2B(9)) \approx 3441.
\]

The right-hand Riemann sum on the interval \([3, 11]\) is given by

\[
9800(2B(5) + 2B(7) + 2B(9) + 2B(11)) \approx 3441.
\]

(Note that the left-hand and the right-hand Riemann sums are actually equal in this case. This makes sense because \(B(3) = B(11) = 0\), and all the other terms are present in both sums.) Both the estimates reach the threshold of strong instability, but neither reach the threshold of extreme instability.

(b) Consider the function appearing in the integrand,

\[
B(h) = \frac{P(h) - A(h)}{A(h) + 273}.
\]

You must use derivatives to justify your answer to the following two questions.

i. Is \(B(h)\) increasing or decreasing on the interval \(3 < h < 7\)?

ii. Is \(B(h)\) increasing or decreasing on the interval \(7 < h < 11\)?

**Solution:** Note that, by the quotient rule, we have

\[
B'(h) = \frac{(P'(h) - A'(h))(A(h) + 273) - (P(h) - A(h))A'(h)}{(A(h) + 273)^2}.
\]

By looking at the graph, we see that \(A(h)\) is always at least \(-30\), so \(A(h) + 273\) is certainly always positive. Also, we see that \(P(h) - A(h) \geq 0\) on the entire interval \([3, 11]\).

Now on the interval \((3, 7)\), we can see from the graph that \(A'(h) < 0\) and that the slope of \(P(h)\) is less negative than the slope of \(A(h)\), so that \(P'(h) > A'(h)\). Combining these signs with those above, we see that the numerator of \(B'(h)\) is of the form \((+)(+) - (+)(-).\)
That is, it must be a positive number minus a negative number, so the numerator is positive. Since the denominator is also positive, $B'(h) > 0$, and $B(h)$ must be increasing on the interval $(3, 7)$.

On the interval $(7, 11)$, now we have that $A'(h) = 0$, but notice that $P'(h)$ is still negative. So the numerator of $B'(h)$ is of the form $(-)(+) - 0$, so is negative. Since the denominator is again positive, $B'(h) < 0$ and $B(h)$ is decreasing on the interval $(7, 11)$.

(c) We would like to compute an overestimate and underestimate of the CAPE value. Recall that strong instability corresponds to $2500 < \text{CAPE} \leq 4000$ and extreme instability corresponds to $\text{CAPE} > 4000$. If you cannot rule out the possibility of extreme instability, you will issue a severe thunderstorm watch.

**Hint:** You will need your answer to part (b).

i. Using left or right Riemann sums (as appropriate) with 4 equal subdivisions, give an overestimate and an underestimate of the integral

$$9800 \int_3^7 \frac{P(h) - A(h)}{A(h) + 273} \, dh.$$  

**Solution:** Since $B(h)$ is increasing on the interval $(3, 7)$, the left-hand Riemann sum will be an underestimate:

$$9800(B(3) + B(4) + B(5) + B(6)) = 9800 \left( 0 + \frac{7.5}{273} + \frac{15}{263} + \frac{17.5}{258} \right) \approx 1493,$$

and the right-hand Riemann sum will be an overestimate:

$$9800(B(4) + B(5) + B(6) + B(7)) = 9800 \left( \frac{7.5}{273} + \frac{15}{263} + \frac{17.5}{258} + \frac{20}{253} \right) \approx 2268.$$

ii. Using left or right Riemann sums (as appropriate) with 4 equal subdivisions, give an overestimate and an underestimate of the integral

$$9800 \int_7^{11} \frac{P(h) - A(h)}{A(h) + 273} \, dh.$$  

**Solution:** Since $B(h)$ is decreasing on the interval $(7, 11)$, the left-hand Riemann sum will be an overestimate:

$$9800(B(8) + B(9) + B(10) + B(11)) = 9800 \left( \frac{15}{253} + \frac{10}{253} + \frac{5}{253} + 0 \right) \approx 1162,$$

and the right-hand Riemann sum will be an underestimate:

$$9800(B(7) + B(8) + B(9) + B(10)) = 9800 \left( \frac{20}{253} + \frac{15}{253} + \frac{10}{253} + \frac{5}{253} \right) \approx 1937.$$
iii. Use your answers to (i) and (ii) to give an overestimate and an underestimate of the CAPE value.

**Solution:** In order to get an underestimate of the integral on the whole interval [3, 11], we combine the two underestimates we found in parts (ii) and (iii) to obtain $1493 + 1162 = 2655$.

Similarly, to get an overestimate of the CAPE we sum the two overestimates to obtain $2268 + 1937 = 4205$.

iv. Can you be certain whether or not the CAPE value reaches the level of strong instability?

**Solution:** From our answer to the previous part, we can be certain that $2655 \leq \text{CAPE} \leq 4205$.

Thus, we can be certain that the CAPE value reaches at least the level of strong instability because it must be greater than 2500.

v. Should you issue the severe thunderstorm watch?

**Solution:** Yes, a thunderstorm warning should be issued as we cannot rule out the possibility of extreme instability based on our calculations. That is, even though we're not sure the CAPE value is greater than 4000, we can't rule it out with the estimates we have.