1. Hakim, the masked dyer of Merv, enjoys collecting wool and dying it purple.
   - Let \( p \) be the amount of purple dye, in milliliters (mL), that Hakim has on hand to dye the wool, and let \( w = f(p) \) be the amount of wool, in kilograms (kg), that Hakim can dye purple when he has \( p \) mL of purple dye.
   - As Hakim dyes the wool, he rates his stress level by measuring the amount of cortisol, a stress hormone, in his blood. Let \( c = g(w) \) be Hakim’s cortisol level, in micrograms per deciliter (\( \mu g/dL \)), when he has dyed \( w \) kg of wool.

   (a) Use a complete sentence to give a practical interpretation of each of the following equations.
      i. \( f(6) = f(5) + 3 \).
      ii. \( g^{-1}(8) = 2 \).
      iii. \( g(f(4)) = 10 \).

   (b) Using the functions \( f \) and \( g \) and/or their inverses, give a mathematical equation representing each of the following sentences.
      i. Hakim can dye the same amount of wool purple whether he has 3 mL of purple dye or 4 mL of purple dye.
      ii. For Hakim to dye 7 kg of wool purple, he needs 6 mL of purple dye.
      iii. When Hakim has 2 mL of purple dye, his cortisol level is ten times as high as it is when he has 7 mL of purple dye.

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Suppose that the graphs of the functions $f$ and $g$, defined on the previous page, are as shown below.

(c) Note that we can express $f(p)$ using the piecewise formula

$$f(p) = \begin{cases} 
1 & \text{for } 2 \leq p < 4 \\
3(p - 4) + 1 & \text{for } 4 \leq p < 6 \\
7 & \text{for } 6 \leq p \leq 7 
\end{cases}$$

Write a piecewise formula for $g(w)$.

(d) Use the formulas from (c) to write a piecewise formula for $c = g(f(p))$, Hakim’s cortisol level, in µg/dL, as a function of his amount $p$ of purple dye, in mL. Then, draw the graph of this function as well.

Now Hakim switches to using rose-colored dye. Let $r$ be the amount of rose-colored dye, in mL, that Hakim has to dye his wool, and let $w = h(r)$ be the amount of wool, in kg, that Hakim can dye when he has $r$ mL of rose-colored dye. The graph of $h$ is shown to the right. Note that the scale on the two axes is not the same.

(e) The function $h$ can be obtained from the function $f$ by one or more graph transformations. Find a formula for $h(r)$ in terms of the function $f$ and $r$.

(f) When he’s too tired to dye wool, Hakim switches to painting masks. The total surface area $A$ of masks that he can paint, in centimeters squared (cm²), is a function of his cortisol level $c$, in µg/dL. Hakim finds that the relationship between $A$ and $c$ is given by the equation

$$A = 2 \left( g^{-1}(c) - 1 \right).$$

Graph the surface area $A$ as a function of his cortisol level $c$.

Note that it will likely be helpful to first draw the graph of $g^{-1}(c)$. 

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2. The city of Labtayt contains a locked tower filled with statues of the city’s past kings. The present-day king enters the tower and examines the statues. He finds that several of them are making a strange humming noise, so he measures the loudness of each sound, in Pascals (Pa).

Note: Throughout this problem, you should give your answers in exact form. For example, $\sqrt{2}$ and $\ln(2)$ are exact values. On the other hand, writing these quantities as 1.41421 and 0.69315 respectively is not exact, because these decimals have been rounded.

(a) The king starts measuring the sound made by the first statue. Initially, it makes $4.8 \cdot 10^{-3}$ Pa of sound; six minutes later, it makes $1.2 \cdot 10^{-3}$ Pa of sound. Let $L(t)$ be the loudness of the first statue, in Pa, $t$ minutes after the king started measuring its sound.

i. Assuming that $L(t)$ is a linear function, find a formula for $L(t)$.

ii. Assuming that $L(t)$ is an exponential function, find a formula for $L(t)$.

iii. What do these two models predict will happen to the loudness in the long run? On the basis of that prediction, which model is more reasonable?

Satisfied in his understanding of the first statue, the king moves on to the next group of statues, a cluster of four. Each statue is labeled with a name, but the statues are so old that the king can only make out the first letter of each name, $H$, $K$, $M$, and $Q$, and decides to refer to each statue by its first letter. Still curious about the strange humming noises, the king measures the loudness of each statue; his four measurements start at the same time. The sound levels, in Pa, are given by the following functions, where e.g. $h(t)$ gives the loudness of statue $H$ at a time $t$ minutes after the measurements begin and similarly for the other functions.

\[
\begin{align*}
h(t) &= 10^{-3} \cdot 5^{-t} \\
m(t) &= (2 \cdot 10^{-5})t \\
k(t) &= 10^{-5} \cdot 2^t \\
q(t) &= (2 \cdot 10^{-5})t + 1.8 \cdot 10^{-4}
\end{align*}
\]

(b) When will statue $H$ be half as loud as statue $K$?

(c) Find a formula for $h_d(t)$, the function that gives the loudness of statue $H$ in decibels at a time $t$ minutes after the king’s measurements begin. Similarly find $k_d$, $m_d$, and $q_d$ for the other three statues as well.

(d) When will statue $K$ be as loud as a jackhammer, 100 dB?
3. The widow Ching, a pirate, is being pursued by Admiral Kwo-Lang down the Zhū Jiāng River. She has constructed a huge water wheel in the river to block his path.

The flow of the river pushes the water wheel, causing it to rotate at a constant speed. The wheel has three blades, each 70 feet long, spaced an even $2\pi/3$ radians apart. One blade is painted red. When each blade is at its lowest point, it just barely scrapes the bottom of the river, which is currently 35 feet deep. The figure below, which is not drawn to scale, shows a vertical cross-section of the wheel and river. In the figure, water flows from left to right, causing the wheel to rotate counterclockwise.

![Diagram of water wheel and river]

During each rotation, the red blade is in the water for 12 seconds. Let $f(t)$ be the height, in feet, of the tip of the red blade above the bottom of the river $t$ seconds after the red blade leaves the surface of the water. That is, the diagram shows the water wheel at $t = 0$.

(a) Determine how long does it takes for the wheel to make one complete revolution. Then give a formula for the function $f(t)$.

The widow Ching later realizes that the admiral will not be stopped by the presence of the wheel alone. As a further countermeasure, she decides to electrify the blades, so that whenever any electrified blade touches the water, the river will course with electricity. At this point, the river has reached its low-tide depth of 15 feet, though the speed of the wheel remains unchanged.

(b) If the widow Ching electrifies only the red blade, for how long is the river electrified during one complete revolution of the wheel at low tide? Give your answer both in exact form and as a decimal approximation.\(^1\)

(c) If instead the widow Ching electrifies all three blades, for how long is the river electrified during one complete revolution of the wheel at low tide? For what percentage of a complete revolution is the river electrified?

\(^1\)It may be very helpful, both for visualizing what you want to find and for verifying your answer, to graph your function $f(t)$ using, for example, www.desmos.com/calculator.