1. 2.71828... Is the Loneliest Number.

a) Bank A offers you the following amazing deal – they will allow you to invest 1 dollar on February 1st, 2019, at an annual growth rate of 100%. Let \( A(t) \) denote the value of your account at Bank A, in dollars, \( t \) years after the time of your initial investment.

i) Write down a formula for \( A(t) \).

**Solution:** The formula for exponential growth will be \( A(t) = ab^t \) where \( a \) is the initial value of the function and \( (b - 1) \) is the growth rate per year. Here we have \( a = 1 \) and \( b - 1 = 1 \), so \( b = 2 \), and so our formula is

\[
A(t) = 2^t.
\]

ii) How much money would be in your account after 1 year?

**Solution:** We use the formula for our function \( A(t) \):

\[
A(1) = 2^1 = 2.
\]

This tells us that after 1 year, the value of the bank account is $2.

iii) What is the continuous growth rate of \( A(t) \)?

**Solution:** The continuous growth rate \( k \) of an exponential function \( A(t) = ab^t \) must satisfy \( e^k = b \). In our case \( b = 2 \), so \( e^k = 2 \), and taking the natural logarithm of both sides of this equation yields \( k = \ln 2 \). We deduce that the continuous growth rate of \( A(t) \) is \( \ln 2 \approx 0.69315 \) or 69.315%.

b) Bank B offers you a slightly different deal, again starting on February 1st, 2019. Instead of increasing your investment by 100% every year, they will increase your investment by \( \frac{100\%}{12} \approx 8.333\% \) every month. This process of breaking up the interest into smaller pieces is called compounding. Let \( B(t) \) represent the value of your account at Bank B, in dollars, \( t \) years after February 1st, 2019 (assuming you accept Bank B’s deal and invest 1 dollar).

i) Write down a formula for \( B(t) \).

**Solution:** We use the formula \( B(t) = ab^t \) of an exponential growth function. We still have \( a = 1 \) since \( a = B(0) \) is the initial value of the bank account. After 1 month, or 1/12 of a year, the account will be worth \( 1 + 1/12 = 13/12 \) dollars. Therefore,

\[
b^{1/12} = \frac{13}{12}.
\]

Solving for \( b \) gives us \( b = (13/12)^{12} \), so

\[
B(t) = \left( \frac{13}{12} \right)^{12t}.
\]
We could have also used approximate values to write

\[ B(t) \approx \left( (1.08333)^{12} \right)^t \approx 2.6129^t. \]

ii) How much money will you have in your Bank B account after 1 year? Is this more than, less than, or the same as, the amount you would have after one year with Bank A? Why?

**Solution:** We compute

\[ B(1) = \left( \frac{13}{12} \right)^{12} \approx 2.6129 \]

So after 1 year, the bank account will have a value of about $2.61. This is more than the value of the bank account after 1 year with Bank A. This is because \( A(t) \) and \( B(t) \) both have the same initial value of \( a = 1 \), but \( B(t) \) has a larger growth factor, hence a larger annual growth rate. Or we can note that with Bank B, starting with the second month we earn interest on our interest.

iii) How long will it take for your initial investment of one dollar to triple in value?

**Solution:** We wonder: for which value of \( t \) is it the case that \( B(t) = 3B(0) \)? In other words, for which value of \( t \) do we have \( ((13/12)^{12})^t = 3 \)? We’d like to solve for \( t \) in this equation. Taking log of both sides of this equation gives us

\[ t \log \left( \frac{13}{12} \right)^{12} = \log(3) \]

Now solving for \( t \) gives us

\[ t = \frac{\log(3)}{\log \left( \frac{13}{12} \right)^{12}} \approx 1.14378 \]

We deduce that after approximately 1.14 years, the value of the bank account will have tripled.
c) Bank C offers you a final deal – they are going to break up and compound your interest even further. Instead of a 100% return on your investment each year like Bank A, or compounding interest approximately 8.333% each month like Bank B, they are going to compound $\frac{100\%}{365} \approx 0.2740\%$ every day. Let $C(t)$ represent the value of your account at Bank C in dollars, $t$ years after February 1st, 2019 (assuming you accept Bank C’s deal and invest 1 dollar).

i) Write down a formula for $C(t)$

**Solution:** After 1 day, or $1/365$ of a year, the account will be worth $1+1/365 = 366/365$ dollars. We still have $a = 1$ as the initial value of the bank account, so again using the general form $C(t) = ab^t$, we get

$$b^{1/365} = \frac{366}{365}.$$ 

Solving for $b$ gives us $b = (366/365)^{365}$. Our function $C(t)$ is given by the formula

$$C(t) = \left(\left(\frac{366}{365}\right)^{365}\right)^t \approx (1.00274)^{365t} \approx 2.71457^t.$$ 

ii) What is the continuous growth rate for $C(t)$?

**Solution:** The continuous growth rate $k$ of an exponential function $C(t) = ab^t$ satisfies $e^k = b$. In our case $b = (366/365)^{365}$, so we need $e^k = (366/365)^{365}$, and taking the natural logarithm of both sides of this equation yields

$$k = \ln\left(\left(\frac{366}{365}\right)^{365}\right) \approx 0.99863.$$ 

We deduce that the continuous growth rate of $C(t)$ is approximately 0.99863 or 99.863%.

iii) How much money will you have in your Bank C account after 1 year?

**Solution:** After 1 year, the value of the bank account is given in dollars by

$$C(1) = \left(\left(\frac{366}{365}\right)^{365}\right)^1 \approx 2.71457,$$

so after 1 year, the account will hold approximately $2.71$. 

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d) Notice that for each of the successive banks, the value of your investment after 1 year is getting closer and closer to $e$ dollars and the continuous growth rate is getting closer and closer to 1. Describe a Bank policy that would make your investment after one year even closer to $e$ than Bank C.\(^1\) (Assume that banks keep track of your exact investment, rather than rounding to the nearest cent.)

**Solution:** We see that as we compound interest more frequently, the value of the account after 1 year approaches $e$ dollars. For this reason, we could get closer to a value of $e$ dollars after 1 year by compounding \(\frac{100\%}{8760}\) \(\approx 0.0114\%\) every hour (there are \(24 \cdot 365 = 8760\) hours in a year). In this case, the function \(D(t)\) which gives the value of the account \(t\) years after the initial investment of 1 is given by

\[
D(t) = \left( \frac{8761}{8760} \right)^{8760}^t
\]

So the value of the account after 1 year is

\[
D(t) = \left( \frac{8761}{8760} \right)^{8760}^1 \approx 2.71813
\]

and this number agrees with \(e = 2.7182\ldots\) up to 3 decimal places. Of course, other reasonable answers are possible here.

\(^1\)Taking this idea to its extreme results in continuously compounded interest.
2. The Odd and Even Couple

Below is a partial graph of an even function $f(x)$ and a table of values for an invertible and odd function $g(x)$. Both $f(x)$ and $g(x)$ have domains $-6 \leq x \leq 6$.

![Graph of $f(x)$ with table of $g(x)$ values]

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-5$</th>
<th>$-2$</th>
<th>$1$</th>
<th>$4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g(x)$</td>
<td>$-9$</td>
<td>$-4$</td>
<td>$1$</td>
<td>$7$</td>
</tr>
</tbody>
</table>

a) Decide whether each of the following expressions exists; if so, find the value, or explain why there is not enough information to find it.

i) $f(g(-2))$

**Solution:** The table yields $f(g(-2)) = f(-4)$. Because $f(x)$ is even, we have

$$f(g(-2)) = f(-4) = f(4) = 2.$$

ii) $g(g^{-1}(0))$

**Solution:** We know that $g(g^{-1}(x)) = x$ as long as $x$ is in the range of $g(x)$. (If $x$ isn’t the range of $g$, then $g^{-1}(x)$ wouldn’t exist). But because zero is in the domain of $g(x)$, and $g(x)$ is odd, we know that we have $g(0) = -g(0)$ so $g(0) = 0$. (You can also think about it graphically – a graph symmetric about the origin must go through the origin.) We can then conclude there is an input value that produces the output zero and

$$g(g^{-1}(0)) = 0.$$

iii) $g(g(0))$

**Solution:** As stated in the previous part, $g(0) = 0$, so

$$g(g(0)) = g(0) = 0.$$ 

iv) $f(g(5))$

**Solution:** Because $g(x)$ is odd, $g(x) = -g(-x)$, and

$$f(g(5)) = f(-g(-5)) = f(9),$$

but 9 is not in the domain of $f(x)$! So this expression does not exist.
v) \( g^{-1}(f(-6) + 1) \)

**Solution:** Since \( f(x) \) is even, \( f(-6) = f(6) = 3 \). So, we have
\[
g^{-1}(f(-6) + 1) = g^{-1}(3 + 1) = g^{-1}(4).
\]
Now because \( g \) is odd, and because \( g(-2) = -4 \), we know that \( g(2) = 4 \). So, \( g^{-1}(f(-6) + 1) = g^{-1}(4) = 2 \).

b) For each of the following functions, carefully draw its entire graph, write a formula for it in terms of \( f(x) \), and find its domain.

i) The function \( r(x) \) obtained from \( f(x) \) by first horizontally stretching (making wider) by a factor of 2 and then shifting to the right 3 units

**Solution:**
After the horizontal stretch have the function \( f \left( \frac{1}{2}x \right) \). Now, be careful: to shift to the right by 3, we need to replace \( x \) with \( (x - 3) \). That is, we want the function \[
\left\{ f \left( \frac{1}{2}(x - 3) \right) \right\}. 
\]
Don’t forget these parentheses. Now, the domain of \( r(x) \) is the domain \([-6, 6]\) of \( f(x) \) stretched to become \([-12, 12]\), and then shifted to the right to become \([-9, 15]\).

![Graph of r(x)](image)

ii) The function \( q(x) \) obtained from \( f(x) \) by first shifting \( f(x) \) to the right 3 units and then horizontally stretching by a factor of 2.

**Solution:**
After the shift to the right we have \( f(x - 3) \). Now, to stretch by a factor of 2, we need to replace \( x \) with \( \frac{1}{2}x \). That is, we want the function \[
\left\{ f \left( \frac{1}{2}(x - 3) \right) \right\}. 
\]
Note that we could put parentheses around the $\frac{1}{2}x$, but, unlike in the previous part, they aren’t required here because of order of operations. Now, the domain of $q(x)$ is the domain $[-6, 6]$ of $f(x)$ shifted to the right to become $[-3, 9]$, and then stretched to become $[-6, 18]$.

\[
y = q(x)
\]

\[
\begin{array}{ccccccccccccccc}
\text{9} & \text{7} & \text{5} & \text{3} & \text{1} & \text{3} & \text{5} & \text{7} & \text{9} & \text{11} & \text{13} & \text{15} & \text{17} & \text{19} \\
\text{3} & \text{2} & \text{1} & \text{1} & \text{2} & \text{3} & \text{2} & \text{1} & \text{1} & \text{2} & \text{3} & \text{2} & \text{1} & \text{1}
\end{array}
\]

iii) Are $r(x)$ and $q(x)$ the same function? Carefully explain why or why not.

**Solution:** They are not the same function! They don’t even have the same domain. There are many ways to explain why the order of transformations matters. One way is to follow the bottom of the “V”. For $r(x)$, first we stretched and then shifted, but because this point started on the $y$-axis, it only moved when we shifted. On the other hand, for $q(x)$, we shifted, which caused the point to move as before, but then, because it was no longer on the $y$-axis, it moved when we stretched too.

c) Is $g(f(x))$ even, odd, neither, or is there not enough information to determine this?

**Solution:** Because $f(x)$ is even, $f(x) = f(-x)$ so

\[g(f(-x)) = g(f(x))\]

and $g(f(x))$ is even.
d) Let \( s(u) = f(e^u) \).

i) What is the largest possible domain of \( s(u) \)?

**Solution:** For \( f(e^u) \) to be defined we need \( e^u \) to be in the domain of \( f(x) \). That is, we need

\[-6 \leq e^u \leq 6.\]

We know \( e^u > 0 \) for any \( u \), so we just need \( e^u \leq 6 \), or equivalently \( u \leq \ln 6 \). The largest possible domain is therefore

\((-\infty, \ln 6]\).

ii) Is the function \( s(u) \) increasing?

**Solution:** On our domain \((-\infty, \ln 6]\), we know \( e^u \) is increasing, and is always between 0 and 6. Now since \( f(x) \) is increasing on \((0, 6]\), we can be sure that the composition \( s(u) = f(e^u) \) is increasing on \((-\infty, \ln 6]\).

iii) Is the function \( s(u) \) invertible?

**Solution:** Because \( s(u) \) is increasing, it must pass the horizontal line test, and so is invertible.
3. It’s Electric

Coulomb’s law describes the electrical force created by two electrically charged objects when they are placed near each other. Coulomb realized that the force $F(d)$, measured in Newtons, depends on the distance $d$, measured in centimeters, between the two charged objects. Specifically Coulomb made the following initial observation: as the distance between two objects increases, the electrical force between them decreases. He aimed to express this physical law $F(d)$ precisely with a formula.

a) What is a reasonable domain for $F(d)$?

**Solution:** A reasonable domain for $F(d)$ would be $[0, \infty)$. The value of $d$ certainly can’t be negative since it doesn’t make sense for two objects to be a negative distance from each other. Also, the value of $d$ can be any positive number since two objects can presumably be arbitrarily far apart.

b) Explain why $F(d)$ should be an invertible function. (Your answer shouldn’t involve any formulas.)

**Solution:** Coulomb observed that as the distance between the two objects increased, the force between them decreased. That means the function $F$ is a decreasing function, and functions which are always increasing or always decreasing throughout their domain are always invertible.

c) Let’s formulate some hypotheses for type of function $F(d)$ might be:

**Hypothesis 1:** $F(d)$ is a linear function

**Hypothesis 2:** $F(d)$ is an exponential function

**Hypothesis 3:** $F(d)$ is inversely proportional to the square of $d$

Coulomb ran an experiment\(^2\) in which he held two charged metallic balls at varying distances from one another and recorded the resulting force at each distance. Below is a table of values you could obtain by running a similar experiment:

<table>
<thead>
<tr>
<th>$d$</th>
<th>1</th>
<th>4</th>
<th>6</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F(d)$</td>
<td>5.31</td>
<td>.139</td>
<td>.05</td>
<td></td>
</tr>
</tbody>
</table>

For each of the hypotheses above, determine whether the hypothesis is consistent with the data. If so, find a formula for $F(d)$ appropriate to that hypothesis. (E.g., find a linear function if Hypothesis 1 is consistent with the data.) Be sure to justify all steps.

**Solution:** We’ll verify or refute each hypothesis separately.

**Hypothesis 1:** To check whether $F$ is linear or not, we need to determine whether the rate of change of $F$ is constant across all the given distance intervals. Let’s compute

the rate of change of $F$ over the intervals $[1, 4]$, $[4, 6]$, and $[6, 10]$.

Rate of change in $[1, 4] = \frac{F(4) - F(1)}{4 - 1} = \frac{0.31 - 5}{4 - 1} \approx -1.56$

Rate of change in $[4, 6] = \frac{F(6) - F(4)}{6 - 4} = \frac{0.139 - 0.31}{6 - 4} = -0.0855$

Rate of change in $[6, 10] = \frac{F(10) - F(6)}{10 - 6} = \frac{0.05 - 0.139}{10 - 6} = -0.0225$

It’s clear that the rates of change in these three intervals aren’t the same, so the function $F(d)$ cannot be linear.

**Hypothesis 2:** To check whether $F$ is exponential, we will check whether the decay factors are the same for each pair of points given.

Decay factor in $[1, 4] : \frac{a^4}{a^1} = \frac{0.31}{5} \Rightarrow a = \left( \frac{0.31}{5} \right)^{1/3} \approx 0.3958$

Decay factor in $[4, 6] : \frac{a^6}{a^4} = \frac{0.139}{0.31} \Rightarrow a = \left( \frac{0.139}{0.31} \right)^{1/2} \approx 0.6696$

Decay factor in $[6, 10] : \frac{a^{10}}{a^6} = \frac{0.05}{0.139} \Rightarrow a = \left( \frac{0.05}{0.139} \right)^{1/4} \approx 0.7744$

It’s clear that the decay rates are not the same, so $F$ cannot be exponential.

**Hypothesis 3:** Suppose $F(d)$ was inversely proportional to the square of $d$. Then $F(d)$ would be given by a formula of the form $\frac{k}{d^2}$ for some constant $k$. Since $F(1) = \frac{k}{1^2} = 5$, the constant would be 5 if $F(d)$ was inversely proportional to $d$. To verify that this could be the case, we need to check $\frac{5}{d^2}$ equals $F(d)$ for the other values of $d$ as well.

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\frac{5}{d^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>0.3125</td>
</tr>
<tr>
<td>6</td>
<td>0.139</td>
</tr>
<tr>
<td>10</td>
<td>0.05</td>
</tr>
</tbody>
</table>

It’s evident from the table that $\frac{5}{d^2}$ agrees with $F(d)$ up to rounding or perhaps measurement error. This shows that Hypothesis 3 is consistent with the collected data.

d) Suppose Coulomb had only collected the first two data points: $(1, 5)$ and $(4, .31)$. Could he have eliminated any of his hypotheses? Your explanation may refer to work done in part (c), but does not need to include formulas as long as your argument is convincing.

**Solution:** No, Coulomb couldn’t have eliminated any of the hypothesis, since it’s possible to find both linear and exponential functions passing through $(1, 5)$ and $(4, 0.31)$ (and since the formula we found for Hypothesis 3 goes through those two points as well). The linear function would be $F(d) \approx -1.56d + 6.56$ (from the rate of change we computed in part (c)), and the exponential function would be $F(d) \approx e^{-0.93d + 2.54} \approx 12.68(0.39)^d$. 

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