1. **Down on the Corner**

Below is the graph of a function $f(x)$.

Some additional information about $f(x)$:

- On the interval $[-2, 0]$, the function is given by $f(x) = -x^2$.
- On the interval $[0, 2]$, the function is given by $f(x) = x^2$.
- $f(x)$ is linear on all other intervals.

a) For what values of $x$ is $f(x)$ not continuous?

**Solution:** The function $f(x)$ is not continuous at $x = -4, 2$ because it has jumps there.

b) For what values of $x$ is $f(x)$ not differentiable?

**Solution:** In addition to the places where it isn’t continuous, $f(x)$ is also not differentiable at corners (and at vertical tangents, though there are not any here). So $f(x)$ is not differentiable at $x = -4, -2, 2, 4$. There is not a corner at the origin, because the slope from both sides approaches zero there.
c) Draw a well-labelled graph of $f'(x)$. Indicate clearly where it is discontinuous or undefined.

**Solution:**

Notice that the values of the horizontal line segments are the values of the slope of the line segments from the graph of $f(x)$, and that the “V” shape comes from the fact that $\frac{d}{dx} \left[ \pm x^2 \right] = \pm 2x$.


d) For what values of $x$ is $f'(x)$ not differentiable?

**Solution:** This is the same question as part (b), but for the graph in part (c). Since $f'(x)$ has a corner at $x = 0$, it is not differentiable there. Note that the second derivative is also not defined at $x = -4, -2, 2, 4$ because those points are not in the domain of $f'(x)$. 
e) Draw a well-labelled graph of $f''(x)$. Indicate clearly where it is discontinuous or undefined.

\[ \begin{array}{c}
\includegraphics[width=\textwidth]{graph.png}
\end{array} \]

**Solution:**
Pay attention to the holes at $x = \pm 4$, since although the slope is 0 on either side of these two points, the derivative is not defined at these points. Note $f''(x)$ is also undefined at 0 and $\pm 2$.

f) Let $g(x) = e^x(f(x))^2$. Calculate $g'(1)$ exactly.

**Solution:** Here we have to use a combination of the product rule and chain rule.

\[ g'(x) = (e^x)' \cdot (f(x))^2 + e^x \cdot ((f(x))^2)' = e^x \cdot (f(x))^2 + e^x \cdot 2f(x) \cdot f'(x), \]

and after simplifying, we get

\[ g'(x) = e^x f(x) \cdot (f(x) + 2f'(x)). \]

Now we can evaluate that

\[ g'(1) = ef(1) \cdot (f(1) + 2f'(1)) = e(1)(1 + 2(2)) = 5e. \]
2. Down on the Farm

As farming practices improve, farmers are able to produce more of a given crop per acre of farmland. A prime example of this is provided by soybeans, for which the yield per acre has been steadily increasing for the past 30 years. Below are some values of the function $Y(t)$, which gives the average yield, measured in bushels per acre, $t$ years after 1988. We also provide some values for the derivative $Y'(t)$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>4</th>
<th>10</th>
<th>16</th>
<th>22</th>
<th>24</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y(t)$</td>
<td>26</td>
<td>37</td>
<td>38</td>
<td>42</td>
<td>44</td>
<td>42</td>
<td>52</td>
</tr>
<tr>
<td>$Y'(t)$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>-1</td>
<td>1.5</td>
<td>-0.5</td>
<td>2.5</td>
</tr>
</tbody>
</table>

The price of soybeans tends to fluctuate from year to year. Let $P(t)$ be the price of one bushel of soybeans, in dollars, $t$ years after 1988. Some values of $P(t)$ and its derivative are shown below.

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>4</th>
<th>10</th>
<th>16</th>
<th>22</th>
<th>24</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(t)$</td>
<td>7.9</td>
<td>5.8</td>
<td>6.6</td>
<td>8.4</td>
<td>9.5</td>
<td>12</td>
<td>9.6</td>
</tr>
<tr>
<td>$P'(t)$</td>
<td>-0.5</td>
<td>0.2</td>
<td>0.8</td>
<td>0.5</td>
<td>-0.1</td>
<td>-0.3</td>
<td>0.4</td>
</tr>
</tbody>
</table>

a) Consider each of the following functions.

i) The annual revenue from one acre of soybeans is given by $A(t) = P(t)Y(t)$. Calculate $A'(4)$. What are the units of $A(t)$?

**Solution:** Since the output of $Y(t)$ is in bushels per acre and the output of $P(t)$ is in dollars per bushel, the output units for $A(t)$ will be dollars per acre. We will use the product rule to differentiate $A(t)$:

$$A(t) = P(t)Y(t)$$
$$A'(t) = \left( P(t)Y(t) \right)'$$
$$= P'(t)Y(t) + P(t)Y'(t)$$
$$\implies A'(4) = P'(4)Y(4) + P(4)Y'(4)$$
$$= (0.2)(37) + (5.8)(2)$$
$$= 19$$
ii) Let \( M(x) = P(x/12) \) give the price of soybeans \( x \) months after 1988. Calculate \( M'(192) \).

**Solution:** We will use the chain rule to differentiate \( M(x) \):

\[
M(x) = P\left(\frac{x}{12}\right) \\
M'(x) = \left(P\left(\frac{x}{12}\right)\right)' \\
= P'\left(\frac{x}{12}\right) \cdot \frac{1}{12} \\
\Rightarrow M'(192) = P'\left(\frac{192}{12}\right) \cdot \frac{1}{12} \\
= P'(16) \cdot \frac{1}{12} \\
= (0.5) \cdot \frac{1}{12} \\
= \frac{1}{24}
\]

iii) Let \( S(t) = \frac{Y(t)}{\cos(2\pi t) + 10} \) be the seasonally adjusted yield. Calculate \( S'(24) \).

**Solution:** We will use the quotient rule to differentiate \( S(t) \):

\[
S(t) = \frac{Y(t)}{\cos(2\pi t) + 10} \\
S'(t) = \frac{(\cos(2\pi t) + 10) \cdot Y'(t) - Y(t) \cdot \frac{d}{dt}(\cos(2\pi t) + 10)}{(\cos(2\pi t) + 10)^2} \\
= \frac{(\cos(2\pi t) + 10) \cdot Y'(t) - Y(t) \cdot ( - \sin(2\pi t) \cdot 2\pi)}{(\cos(2\pi t) + 10)^2} \\
\Rightarrow \frac{dS}{dt}(24) = \frac{(\cos(2\pi(24)) + 10) \cdot Y'(24) - Y(24) \cdot \left( - \sin(2\pi(24)) \cdot 2\pi \right)}{(\cos(2\pi(24)) + 10)^2} \\
= \frac{(1 + 10) \cdot Y'(24) - Y(24) \cdot 0}{(1 + 10)^2} \\
= \frac{\frac{dy}{dt}(24)}{11} \\
= \frac{(-0.5)}{11} = -\frac{1}{22}
\]
b) Estimate each of the following values.

i) $P''(7)$

**Solution:** $P''(7)$ can be best approximated by finding the average rate of change of $P'$ between the $t$-values nearest to 7:

\[
P''(7) \approx \frac{P'(10) - P'(4)}{10 - 4} = \frac{(0.8) - (0.2)}{6} = 0.1
\]

ii) $\frac{d}{dt}(P'(t)Y(t))$ at $t = 30$

**Solution:** We’ll once again use the product rule and then approximate $P''$ via the average rate of change of $P'$ with the pair of datapoints nearest to $t = 30$:

\[
\left. \frac{d}{dt}(P'(t)Y(t)) \right|_{t=30} = \left[ P''(t)Y(t) + P'(t)Y'(t) \right]_{t=30} \\
= P''(30)Y(30) + P'(30)Y'(30) \\
\approx \left( \frac{P'(30) - P'(24)}{30 - 24} \right)Y(30) + P'(30)Y'(30) \\
= \left( \frac{(0.4) - (-0.3)}{30 - 24} \right)(52) + (0.4)(2.5) \\
= 7.0667
\]

iii) $\frac{d}{dt}(e^{Y'(t)})$ at $t = 0$

**Solution:** This time, we’ll use the chain rule and then approximate $Y''$ as we did with $P''$ previously:

\[
\left. \frac{d}{dt}(e^{Y'(t)}) \right|_{t=0} = \left[ e^{Y'(t)} \cdot Y''(t) \right]_{t=0} \\
= e^{Y'(0)} \cdot Y''(0) \\
\approx e^{Y'(0)} \left( \frac{Y'(4) - Y'(0)}{4 - 0} \right) \\
= e(\frac{(2) - (1)}{4 - 0}) \\
= \frac{e}{4} \approx 0.6796
\]
3. Constants are Changing

Consider the function $h(x)$ where $k$ and $B$ are constants:

$$h(x) = \begin{cases} 
  x^2 - Bx + 1 & x \leq k \\
  -6x & x > k.
\end{cases}$$

Note that each part below is a separate problem.

a) Find all possible values of $B$ and $k$ such that $h(2) = 0$.

**Solution:** The formula we use to evaluate $h(2)$ will depend on whether $2 \leq k$ or $2 > k$. If $2 > k$, then $h(2) = -6(2) = -12$, while if $2 \leq k$, then

$$h(2) = 2^2 - B(2) + 1 = 4 - 2B + 1 = 5 - 2B.$$ 

If $h(2) = 0$, then we can rule out the case where $2 > k$, so instead we know that $2 \leq k$ and

$$h(2) = 5 - 2B = 0 \implies B = \frac{5}{2}.$$ 

Whenever $k \geq 2$ and $B = 5/2$ we have $h(2) = 5 - 2(5/2) = 0$, so the solution is $k \geq 2$ and $B = \frac{5}{2}$.

b) Find all possible values of $B$ and $k$ such that $h(0) = 2$.

**Solution:** As in part (a), we know that $h(0) = -6(0) = 0$ if $0 > k$, and

$$h(0) = 0^2 - B(0) + 1 = 1$$

if $0 \leq k$. Therefore there are no possible values of $B$ and $k$ such that $h(0) = 2$.

c) Find all possible values of $B$ and $k$ such that $h'(1) = 5$.

**Solution:** Using the power rule,

$$h'(x) = \begin{cases} 
  2x - B & x < k \\
  -6 & x > k.
\end{cases}$$

Note that we’re not sure that $h$ is differentiable at $k$, so we shouldn’t use the condition $x \leq k$ in the first part of the function.

If $1 > k$, we have $h'(1) = -6$, not 5. If $k = 1$, we will either have $h'(1)$ DNE, or, if it does exist, $h'(1) = -6$ again, because the slopes from both sides would have to match. So, to have $h'(1) = 5$, we will need $1 < k$ and

$$h'(1) = 2(1) - B = 2 - B = 5 \implies B = -3.$$ 

Therefore, the solution is $k > 1$ and $B = -3$. 

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d) Find all possible values of $B$ and $k$ such that $h(x)$ is decreasing at $x = 0$.

**Solution:** If $h(x)$ is decreasing at $x = 0$, then $h'(0) \leq 0$, or it could be that $h(x)$ is not differentiable at $x = 0$. Recall that

$$h'(x) = \begin{cases} 2x - B & x < k \\ -6 & x > k \end{cases}$$

If $0 < k$, then

$$h'(0) = 2(0) - B = -B$$

so we would need $-B \leq 0 \implies B \geq 0$. If $B > 0$ then $h'(0) < 0$ so $h(x)$ is decreasing at $x = 0$; however if $B = 0$ then $h(x)$ is not decreasing at 0, since then $x = 0$ appears on the graph of $h(x)$ as the vertex of the parabola $x^2 + 1$. So $k > 0$ and $B > 0$ is one possibility.

Next, if $k < 0$, then $h'(0) = -6$ so $h(x)$ is decreasing at $x = 0$ regardless of the value of $B$. So $k < 0$ is another possibility.

Finally, if $k = 0$, then

$$\lim_{x \to 0^-} h(x) = 0^2 - B(0) + 1 = 1$$

and

$$\lim_{x \to 0^+} h(x) = -6(0) = 0,$$

so the graph is not continuous at 0. Moreover, $h(x)$ decreases to the right of $x = 0$, since $h'(x) = -6 < 0$ for $x > 0$. For $h(x)$ to be decreasing at $x = 0$, we need it to be decreasing to the left of $x = 0$ too; this will be the case if $h'(x) = 2x - B < 0$ for all $x < 0$, i.e. if $-B \leq 0 \iff B \geq 0$.

Therefore, the last possibility is $k = 0$ and $B \geq 0$.

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e) Find all possible values of $B$ and $k$ such that the function $h(x)$ is continuous.

**Solution:** Note that $h(x)$ is continuous for $x > k$ and $x < k$, since it is given by polynomials in both cases. At $x = k$, $h(x)$ will be continuous if and only if the left- and right-hand limits of $h(x)$ agree. Setting the two formulas in the piecewise definition of $h(x)$ equal to each other for $x = k$, we have

$$k^2 - Bk + 1 = -6k.$$

Solving for $B$, we can write the solution as $B = (k^2 + 6k + 1)/k$ and $k \neq 0$. If you solve for $k$ instead, using the quadratic formula, you will find that $k = \frac{-6 \pm \sqrt{(6-6)^2-4}}{2}$, but note that in order to ensure the quantity under the square root is nonnegative, we need $B \leq 4$ or $B \geq 8$. So an equivalent solution is $k = \frac{-6 \pm \sqrt{(6-6)^2-4}}{2}$, and $B \leq 4$ or $B \geq 8$.

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f) Find all possible values of $B$ and $k$ such that the function $h(x)$ is differentiable.

**Solution:** If $h(x)$ is differentiable then $h(x)$ is also continuous, so it must be the case that $B = (k^2 + 6k + 1)/k$. In addition, the left- and right-hand limits of $h'(x)$ at $x = k$ must agree for $h(x)$ to be differentiable at $x = k$, so we have

$$2k - B = -6 \iff B = 2k + 6.$$
Setting the two expressions for $B$ equal to each other allows us to solve for $k$:

$$2k + 6 = \frac{(k^2 + 6k + 1)}{k} \iff 2k^2 + 6k = k^2 + 6k + 1 \iff k^2 = 1 \iff k = \pm 1.$$ 

If $k = 1$ then $B = 8$, and if $k = -1$ then $B = 4$. Therefore, the solutions are $k = 1$ and $B = 8$, or $k = -1$ and $B = 4$. 

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