1. Moving On Up

a) Implicit functions often arise in multivariable calculus when there is a third quantity determined by both $x$ and $y$ values. One example of this is a topographic map (see image below at left), which shows the elevation of an area through a series of curves; every point on a given curve has the same elevation. In this way topographic maps are able to inform park rangers, hikers, and other outdoor enthusiasts about the elevation features of the area. In the first part of this problem we’re going to look at a single elevation curve in isolation.

The figure above at right is the graph of the curve $(x^2 + y^2)^2 + c(x^2 - y^2) = 7$, where $c$ is a constant. This curve represents all the locations on a mountain that are at an elevation of 7 thousand feet above sea level. (Note this is not the same mountain as the one shown in the map to the left.) The coordinate $(x, y)$ represents a location $x$ miles east and $y$ miles north of the mountain’s peak. Note that the curve passes through the point $(2, 1)$.

i) Find $c$.

**Solution:** Since the curve passes through the point $(2, 1)$, the equation $(x^2 + y^2)^2 + c(x^2 - y^2) = 7$ must be satisfied when both $x = 2$ and $y = 1$:

$$(2^2 + 1^2)^2 + c(2^2 - 1^2) = 7$$

$$25 + 3c = 7$$

$$c = -6.$$ 

The solution is $c = -6$.

ii) Find a formula for $\frac{dy}{dx}$ in terms of $x$ and $y$.

**Solution:** We use the method of implicit differentiation to find $\frac{dy}{dx}$ treating $y$ as a function of $x$. 

First differentiate the left side of the equation \((x^2 + y^2)^2 - 6(x^2 - y^2) = 7\) with respect to \(x\):

\[
\frac{d}{dx} ((x^2 + y^2)^2 - 6(x^2 - y^2)) = 2(2x + 2y \frac{dy}{dx}) - 12x + 12y \frac{dy}{dx} = (4y(x^2 + y^2) + 12y) \frac{dy}{dx} - (12x - 4x(x^2 + y^2)).
\]

Since the right side of the equation is a constant, its derivative with respect to \(x\) is zero. Setting the derivative of the left and right side equal to each other yields

\[
(4y(x^2 + y^2) + 12y) \frac{dy}{dx} - (12x - 4x(x^2 + y^2)) = 0
\]

\[
\frac{dy}{dx} = \frac{x(3 - x^2 - y^2)}{y(3 + x^2 + y^2)}
\]

iii) Find the equation of the tangent line to the curve at \((2,1)\).

**Solution:** The slope, \(m\), of the tangent line to the curve is given by the derivative \(\frac{dy}{dx}\) at the point \((2,1)\). Using the formula derived in part (ii), we have

\[
m = \frac{2(3 - 2^2 - 1^2)}{1(3 + 2^2 + 1^2)} = \frac{-4}{8} = -\frac{1}{2}.
\]

Since the tangent line passes through the point \((2,1)\), it follows by the point-slope form for a line that the tangent line to the curve at \((2,1)\) is given by

\[
y = 1 - \frac{1}{2}(x - 2).
\]

iv) Suppose your friend John hikes around the mountain along this curve in a clockwise direction. There is exactly one place where he is travelling due south. Find its coordinates. 

*Hints: You can probably guess the \(y\)-coordinate of this location, but make sure you verify that the curve has the appropriate slope at this point. Also, it may be helpful to note that an expression like \(z^4 + 2z^2 - 3\) can be factored into \((z^2 - 1)(z^2 + 3)\)*

**Solution:** The direction John is travelling is given by the direction of the tangent line to the curve at John’s location. Since \((x, y)\) represents the location \(x\) miles east and \(y\) miles north of the mountain’s peak, when John is travelling due south, the tangent line to the curve (his path) must be vertical. The tangent line to the curve is vertical if the denominator of \(\frac{dy}{dx}\) is 0, i.e. if \(y(3 + x^2 + y^2) = 0\). Since \(y(3 + x^2 + y^2)\) will equal 0 when \(y = 0\). The other factor, \((3 + x^2 + y^2)\) is never zero since \(3 > 0, x^2 \geq 0\) and \(y^2 \geq 0\). We can also see in the graph above that the only vertical lines happen when \(y = 0\).
Plugging in \( y = 0 \) into the equation for the curve gives
\[
x^4 - 6x^2 = 7
\]
\[
x^4 - 6x^2 - 7 = 0
\]
\[
(x^2 - 7)(x^2 + 1) = 0
\]
\[
x = \pm \sqrt{7}.
\]

Hence, the only two points on the curve at which there is a vertical tangent line are \((-\sqrt{7}, 0)\) and \((\sqrt{7}, 0)\).

Looking at the plot of the curve shown above, we can observe that at \((-\sqrt{7}, 0)\) John is travelling north and at \((\sqrt{7}, 0)\) John is travelling due south because he is traversing the curve clockwise. When John is \(\sqrt{7}\) miles east of the mountain’s peak, he is travelling due south.

v) There will also be several places where John is travelling due west. Find the coordinates for one of these locations.

**Solution:** When John is travelling due west, the tangent line to the curve (his path) must be horizontal, i.e. \( \frac{dy}{dx} = 0 \) which is the case if and only if
\[
x(3 - x^2 - y^2) = 0
\]
\[
x = 0 \text{ or } (3 - x^2 - y^2) = 0.
\]

Plugging in \( x = 0 \) in to the equation for the curve or alternatively looking at the plot of the curve, the only two with \( x \)-coordinate equal to 0 on the curve are \((0, 1)\) and \((0, -1)\). Since John is hiking the curve clockwise, he is travelling due east at \((0, 1)\) and due west at \((0, -1)\). When John is 1 mile south of the mountain’s peak, he is travelling due west. The other two points when John is travelling due west are \((\pm \sqrt{\frac{5}{3}}, -\frac{2}{\sqrt{3}})\).

b) Next, we’ll use an implicit function for a more strictly mathematical application. The function
\[
y = x^x,
\]
with domain \( x > 0 \), is neither a power function (since it doesn’t have a constant exponent) nor an exponential function (since it doesn’t have a constant base). It’s actually called a “power tower” function, and none of the Chapter 3 rules on their own will let us differentiate it. However, taking the natural log and applying a log rule gives us the implicit relationship
\[
\ln(y) = x \ln(x).
\]

i) Use implicit differentiation to find a formula for \( \frac{dy}{dx} \) in terms of \( x \) and \( y \).

**Solution:** Treating \( y \) as function of \( x \) and differentiating both sides with respect to \( x \) yields
\[
\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(x \ln(x)) \implies \frac{dy}{dx} \cdot \frac{1}{y} = \ln(x) + x \cdot \frac{1}{x} \implies \frac{dy}{dx} = y(\ln(x) + 1).
\]
ii) Write $\frac{dy}{dx}$ in terms of $x$ only. You’ll need to use the original formula $y = x^x$ and your answer to (i).

**Solution:** Plugging $y = x^x$ back into the equation for $\frac{dy}{dx}$ derived in (i), we get

$$\frac{dy}{dx} = x^x(\ln(x) + 1).$$

iii) Consider the function $\sin(x)^{\sin(x)}$ on the domain $(0, \pi)$. Find its derivative.

*Hint: use ii. and the chain rule.*

**Solution:** Let $f(u) = u^u$ and $g(x) = \sin(x)$, then $f(g(x)) = \sin(x)^{\sin(x)}$ using the chain rule and (ii) then yields

$$\frac{d}{dx} \left( \sin(x)^{\sin(x)} \right) = f'(\sin(x)) \cdot g'(x) = \sin(x)^{\sin(x)} (\ln(\sin(x)) + 1) \cos(x).$$

Thus,

$$\frac{d}{dx} \left( \sin(x)^{\sin(x)} \right) = \sin(x)^{\sin(x)} (\ln(\sin(x)) + 1) \cos(x).$$

2. **Are You Smarter Than Your Calculator?**

For parts (a) and (b), you should only use your calculator for very simple arithmetic that you could do by hand. In particular, make sure you use calculus, and not your calculator, to decide whether your linear approximations are overestimates or underestimates.

(a) Let’s approximate $\cos(0.1)$.

i) Write a formula for the linear approximation of $\cos(x)$ at $x = 0$.

**Solution:** The general formula for the linear approximation of a function, $f(x)$, near an input, $x = a$, is $L(x) = f(a) + f'(a)(x - a)$. In this part of the problem we have that $f(x) = \cos(x)$ and $a = 0$. First we find that $f(0) = \cos(0) = 1$. Next, since $f'(x) = -\sin(x)$, we find that $f'(0) = -\sin(0) = 0$. Thus, we have that $L(x) = 1 + 0(x - 0)$, so $L(x) = 1$.

ii) Use this linear approximation to estimate $\cos(0.1)$.

**Solution:** Since $f(x) \approx L(x)$ near $x = 0$, we find that $f(0.1) = \cos(0.1) \approx L(0.1) = 1$.

iii) Is the value you found an overestimate or underestimate of the actual value of $\cos(0.1)$?

**Solution:** We can determine this by checking the concavity of $f(x) = \cos(x)$ near $x = 0$. The second derivative is $f''(x) = -\cos(x)$, and since $f''(0) = -\cos(0) = -1 < 0$, this means that $f(x)$ is concave down near $x = 0$ and therefore our answer from part (ii) is an overestimate.
iv) Write a formula for the quadratic approximation of \( \cos(x) \) at \( x = 0 \).

**Solution:** The general formula for the quadratic approximation of a function, \( f(x) \), near an input \( x = a \), is 
\[
Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2.
\]
We already found in part (i) that \( f(0) = 1 \) and \( f'(0) = 0 \), and in part (iii) we found that \( f''(0) = -1 \). Thus, we find that:
\[
Q(x) = 1 - 0(x-0) + \frac{-1}{2}(x-0)^2.
\]

v) Use this quadratic approximation to estimate \( \cos(0.1) \).

**Solution:** Since \( f(x) \approx Q(x) \) near \( x = 0 \), we find that 
\[
f(0.1) = \cos(0.1) \approx Q(0.1) = 1 - \frac{1}{2}(0.1)^2 = 0.995.
\]

(b) Now let’s approximate \( \ln(0.9) \).

i) Write a formula for the linear approximation of \( \ln(x) \) at \( x = 1 \).

**Solution:** Now we have the function \( g(x) = \ln(x) \) and \( a = 1 \), so we can approximate \( g(x) \) near \( x = 1 \) using the linear approximation 
\[
L(x) = g(1) + g'(1)(x-1).
\]
Since \( g(x) = \ln(x) \), we find that \( g(1) = \ln(1) = 0 \). Next, since \( g'(x) = \frac{1}{x} \), we have \( g'(1) = \frac{1}{1} = 1 \). So the linear approximation of \( g(x) = \ln(x) \) near \( x = 1 \) is 
\[
L(x) = 0 + 1(x-1) = x - 1.
\]

ii) Use this linear approximation to estimate \( \ln(0.9) \).

**Solution:** Since \( g(x) \approx L(x) \) near \( x = 1 \), we find that:
\[
g(0.9) = \ln(0.9) \approx L(0.9) = 0.9 - 1 = -0.1.
\]

iii) Is the value you found an overestimate or underestimate of the actual value of \( \ln(0.9) \)?

**Solution:** Just like we did in part (a)(iii), we will use the second derivative to determine this. Since \( g'(x) = \frac{1}{x} \), using the power rule, we find that \( g''(x) = -\frac{1}{x^2} \). This means that \( g''(1) = -\frac{1}{1^2} = -1 < 0 \), so \( g(x) \) is concave down near \( x = 1 \) and therefore our answer from part (ii) of this part of the problem is an overestimate.

iv) Write a formula for the quadratic approximation of \( \ln(x) \) at \( x = 1 \).

**Solution:** We already know from part (i) that \( g(1) = 0 \) and \( g'(1) = 1 \), and from part (iii) we found that \( g''(1) = -1 \). Putting this together with the general formula for a quadratic approximation, we find that 
\[
Q(x) = 0 + 1(x-1) + \frac{(-1)}{2}(x-1)^2
\]
so 
\[
Q(x) = (x-1) - \frac{1}{2}(x-1)^2.
\]

v) Use this quadratic approximation to estimate \( \ln(0.9) \).

**Solution:** Since \( g(x) \approx Q(x) \) near \( x = 1 \), we find that:
\[
g(0.9) = \ln(0.9) \approx Q(0.9) = (0.9 - 1) - \frac{1}{2}(0.9 - 1)^2 = -0.1 - \frac{1}{2}(0.01) = -0.105.
\]
(c) Now, use your calculator to find $\cos(0.1)$ and $\ln(0.9)$. How many decimal places does your calculator give? What do you think your calculator does to obtain these better approximations?

**Solution:** Using our calculator we see that $\cos(0.1) \approx 0.9950041653$ and $\ln(0.9) \approx -0.1053605157$. The TI-84, at least, gives an estimate out to 10 decimal places for each calculation. Our linear approximations were accurate to about 1 decimal place, and our quadratic approximations were accurate to 3 decimal places. If instead of using linear or quadratic functions we approximated $\cos(x)$ and $\ln(x)$ with cubics, quartics, or even higher order polynomials, then we could get even more accurate answers.

### 3. Prime Time

Rare earth metals are increasingly important in the manufacturing of batteries, electronic devices, and various other products. As a rare earth metal is mined, its reserves are depleted and its price rises. Let $p(x)$ be the price, in dollars per pound, of a particular rare earth metal, when the earth’s reserves of the metal are estimated to be $x$ million tons.

It is known that $p(x)$ is differentiable, decreasing, and concave up on its domain of $(0, \infty)$. Some values of $p(x)$ and $p'(x)$ are given below.

<table>
<thead>
<tr>
<th>$x$</th>
<th>5</th>
<th>8</th>
<th>9.4</th>
<th>10.1</th>
<th>18.2</th>
<th>21.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(x)$</td>
<td>412</td>
<td>72.8</td>
<td>43.3</td>
<td>35</td>
<td>10.1</td>
<td>8</td>
</tr>
<tr>
<td>$p'(x)$</td>
<td>-320</td>
<td>-30.8</td>
<td>-14</td>
<td>-8</td>
<td>-0.82</td>
<td>-0.45</td>
</tr>
</tbody>
</table>

(a) Give practical interpretations of the following quantities. Make sure your statements make sense in the context of this problem.

i) $p(8) = 72.8$

**Solution:** When the earth’s reserves of this particular rare earth metal are estimated to be 8 million tons, its price is $72.8 per pound.

ii) $p'(8) = -30.8$

**Solution:** When the estimated earth’s reserves of this particular rare earth metal **decrease from 8 million tons to 7.9 million tons**, its price will **increase approximately $3.08 per pound**. Note that it doesn’t make sense to talk about reserves increasing in the context of this problem!

(b) Consider the inverse function $p^{-1}(x)$.

i) Explain how we can be sure that $p(x)$ is invertible.

**Solution:** The problem states that $p(x)$ is known to be decreasing. Therefore it must be invertible.

ii) Is $p^{-1}(x)$ increasing, decreasing, or neither? Explain.

**Solution:** $p^{-1}(x)$ is decreasing. This is because the inverse function of a decreasing function must be decreasing. There are a number of ways to think through and justify this fact. One way is to draw a decreasing function’s graph, reflect it over the line $y = x$ (which gives the inverse function’s graph), and see that it is still a decreasing function. You could also describe what’s happening in words.
iii) Find \((p^{-1})'(8)\). What are the units on this quantity? Give a practical interpretation.

**Solution:**
\[
(p^{-1})'(8) = \frac{1}{p'(p^{-1}(8))} = \frac{1}{p'(21.6)} = \frac{1}{-0.45} = \frac{20}{9}.
\]
The units on this quantity are **millions of tons per dollars per pound**. In this practical setting, it means that when the price of this particular rare earth metal **increases from $8 per pound to $8.45 per pound**, its estimated reserves on earth have **decreased by approximately 1 million tons**.

Note that we didn’t have to use “increases from $8 per pound to $8.45 per pound.” This increment just works out nicely because the quantity above is equal to \(-1/0.45\). The important part is that the inverse derivative relates a change in millions of tons in reserve to a price in dollars per pound.

(c) Now consider the derivative function \(p'(x)\).

i) Is \(p'(x)\) increasing, decreasing, or neither? Explain.

**Solution:** According to the problem, \(p(x)\) is differentiable and concave up on its domain \((0, \infty)\), which implies \(p'(x)\) is increasing on \((0, \infty)\).

ii) Is \(p'(x)\) invertible? Explain.

**Solution:** Since \(p'(x)\) is increasing, it is invertible. This follows by the same reasoning as (b)(i).

iii) Find \((p')^{-1}(-8)\). What are the units on this quantity?

**Solution:** Since
\[p'(10.1) = -8,
\]
we have
\[(p')^{-1}(-8) = 10.1.
\]
This quantity has unit **million tons**.

iv) Rewrite your answer to iii. as a statement of the form \(p'(?) = ?\). Then give a practical interpretation.

**Solution:** By the definition of inverse functions, we have
\[p'(10.1) = -8.
\]
An example practical interpretation would be:
When the estimated earth’s reserves of this particular rare earth metal **decreases from 10.1 million tons to 10 million tons**, its price will **increase approximately $0.8 per pound**.