STUDENT SOLUTIONS

1. With my Family

Consider the family of functions \( f(x) = ax^2 e^{-bx} \) where \( a \) and \( b \) are constants.

You may find desmos.com very helpful for understanding the following problems, but you must justify all your answers using the methods you learned in class.

a) Assuming \( a \) and \( b \) are not zero (only for this part), find the \( x \) and \( y \) coordinates of all critical points of \( f(x) \). Your answers may be in terms of \( a \) and/or \( b \).

Solution: We will use the product rule to differentiate \( f(x) = ax^2 e^{-bx} \):

\[
f'(x) = \frac{d}{dx}(ax^2) e^{-bx} + ax^2 \frac{d}{dx}(e^{-bx})
\]

\[
\implies f'(x) = 2axe^{-bx} - abx^2 e^{-bx}
\]

\[
= ax(2 - bx)e^{-bx}.
\]

Note that \( f'(x) \) is defined on \((-\infty, \infty)\), so there are no places where \( f'(x) \) DNE. Also, \( f'(x) = ax(2 - bx)e^{-bx} = 0 \) at \( x = 0 \) and \( x = 2/b \).

Therefore, if \( a \) and \( b \) are both non-zero, the function has two critical points at \( x = 0 \) and \( x = 2/b \).

The \( y \) coordinates at these two points are:

\[
f(0) = 0
\]

\[
f(2/b) = a \left(\frac{2}{b}\right)^2 e^{-b(2/b)} = \frac{4a}{b^2} e^{-2}
\]

b) Find all possible values\(^1\) of \( a \) and \( b \) such that \( f(x) \) has only one critical point.

Solution:

i. When \( a = 0 \), \( f(x) = 0 \). Then, \( f'(x) = 0 \) for all \( x \) on \((-\infty, \infty)\).

ii. When \( a \neq 0 \), \( b = 0 \), \( f(x) = ax^2 \). Then, \( f'(x) = 2ax \), and the function has only one critical point at \( x = 0 \).

iii. When \( a \neq 0 \), \( b \neq 0 \), by the discussion in part (a) of the problem, we know that the function has two critical points at \( x = 0 \) and \( x = 2/b \).

Therefore, the function has only one critical point when \( a \neq 0 \) and \( b = 0 \).

\(^1\)See Problem 3 on Team HW #4 for a discussion about what this means.
c) Is it possible for $f(x)$ to have more than two critical points?

**Solution:** According to part (b), situation i, when $a = 0$, $f(x)$ is constant, so $f'(x) = 0$ everywhere. By our definition of critical points, every point on $f(x)$ is a critical point. Therefore, $f(x)$ can have more than two critical points when $a = 0$.

d) Find all possible values of $a$ and $b$ such that $f(x)$ has a local minimum on the interval $(0, \infty)$.

**Solution:** We again consider different possible cases, as in part (b).

i. When $a = 0$, the function $f(x)$ is constant. By definition, every point on the function is a local max and a local min.

ii. When $a \neq 0$, $b = 0$, the function has only one critical point at $x = 0$. Therefore, there can’t be a local minimum on $(0, \infty)$.

iii. When $a \neq 0$, $b \neq 0$, the function has two critical points at $x = 0$ and $x = 2/b$.

Now, if $b < 0$, we have $2/b < 0$. This means there can’t be a local minimum on $(0, \infty)$.

On the other hand, if $b > 0$, we have $2/b > 0$, so there is a critical point in $(0, \infty)$. We test whether it is a local minimum using the second derivative test.

$$f''(x) = \frac{d}{dx} f'(x)$$
$$= \frac{d}{dx} \left(2ax - abx^2\right)e^{-bx}$$
$$= (2a - 2abx)e^{-bx} - b(2ax - abx^2)e^{-bx}$$
$$= ae^{-bx}(2 - 4bx + b^2x^2)$$

$$f''(2/b) = ae^{-b(2/b)} \left(2 - 4b \left(\frac{2}{b}\right) + b^2 \left(\frac{2}{b}\right)^2\right)$$
$$= ae^{-2} \left(2 - 8 + 4\right)$$
$$= -2ae^{-2}$$

So our answer will depend on $a$. If $a < 0$, we have $f''(x) > 0$, so in this case, the critical point at $x = 2/b$ is a local minimum.

If $a > 0$, we have $f''(x) < 0$, so in that case, $x = 2/b$ is a local maximum.

Therefore, summing up, when $a = 0$ and $b$ is any value, or when $a < 0$ and $b > 0$, the function $f(x)$ has a local minimum on $(0, \infty)$.
e) In this last part we will characterize global maxima in a variety of situations.
   i. If $a > 0$ and $b > 0$, will $f(x)$ have a global maximum on $[0, \infty)$? If so, find its $x$ and $y$
coordinates. If not, explain.
   ii. If $a > 0$ and $b < 0$, will $f(x)$ have a global maximum on $[0, \infty)$? If so, find its $x$ and $y$
coordinates. If not, explain.
   iii. If $a < 0$ and $b > 0$, will $f(x)$ have a global maximum on $[0, \infty)$? If so, find its $x$ and $y$
coordinates. If not, explain.
   iv. If $a < 0$ and $b < 0$, will $f(x)$ have a global maximum on $[0, \infty)$? If so, find its $x$ and $y$
coordinates. If not, explain.

Solution:
   i. If $a > 0$, $b > 0$, the critical points $x = 0$ and $x = 2/b$ are both in $[0, \infty)$. Also, $x = 0$
is also an endpoint, and $f(0) = 0$. By the result of part (a),
   \[
f(2/b) = \frac{4a}{b^2} e^{-2} > 0.
   \]
Finally, we also need to consider what happens when $x$ goes to the positive infinity:
   \[
   \lim_{x \to \infty} f(x) = 0.
   \]
Therefore, the function has a global maximum on $[0, \infty)$ at $x = 2/b$, and
   \[
f(2/b) = \frac{4a}{b^2} e^{-2}.
   \]
   ii. If $a > 0$, $b < 0$, the critical point $2/b$ is no longer on this interval. We still have
   $f(0) = 0$, but now, when $x$ goes to positive infinity we have
   \[
   \lim_{x \to \infty} f(x) = \infty.
   \]
Therefore, the function doesn’t have a global maximum on $[0, \infty)$.
   iii. If $a < 0$, $b > 0$, the function $f(x)$ has critical points $x = 0$ and $x = 2/b$ on $[0, \infty)$. We have
   $f(0) = 0$, and by the result of part (a),
   \[
f(2/b) = \frac{4a}{b^2} e^{-2} < 0.
   \]
When $x$ goes to positive infinity,
   \[
   \lim_{x \to \infty} f(x) = 0.
   \]
Therefore, the function has a global maximum on $[0, \infty)$ at $x = 0$, and $f(0) = 0$.
   iv. If $a < 0$, $b < 0$, the critical point $2/b$ is no longer on this interval. We still have
   $f(0) = 0$, but now when $x$ goes to positive infinity we have
   \[
   \lim_{x \to \infty} f(x) = -\infty.
   \]
Therefore, the function has a global maximum on $[0, \infty)$ at $x = 0$, and $f(0) = 0$. 
Arun, a fictional smallholder farmer in India, is deciding on how many cassava roots to plant at the beginning of the growing season. He has the room plant as many as 450 cassava roots, but isn’t sure this is the best choice to maximize his profit.

a) Arun knows that he will be able to harvest 8 out of every 10 of the cassava that he plants, since some will be damaged by pests and weather. At the local market, he can sell up to 240 cassava roots for a price of 2000 rupees per root to individuals. After this, he will sell the remaining number of roots to wholesalers at the market for 1500 rupees per root.

i) If Arun decides to plant 300 cassava roots, how many would he end up bringing to market? What would his revenue be in this case?

**Solution:** If Arun plants 300 roots, then \(300 \times \frac{8}{10} = 240\) of them will be harvested. Consequently, Arun will sell his entire harvest to individuals. His total revenue will then be \(240 \times 2000 = 480,000\) rupees.

ii) If Arun decides to plant 400 cassava roots, how many would he end up bringing to market? What would his revenue be in this case?

**Solution:** If Arun plants 400 roots, then \(400 \times \frac{8}{10} = 320\) of them will be harvested. Consequently, Arun will sell 240 roots to individuals and 80 roots to wholesalers at the market. His revenue will be \(240 \times 2000 = 480,000\) rupees from the individual sellers and \(80 \times 1500 = 120,000\) rupees from the wholesalers, giving him \(600,000\) rupees in total.

iii) Write a general formula for \(R(k)\), the revenue, in hundreds of rupees, Arun will receive at market if he plants \(k\) cassava roots at the start of the growing season. Note that \(R(k)\) will be a piecewise function. Use your answers from part (i) and (ii) to check your formula.

**Solution:**
If \(k \leq 300\), then \(k \times \frac{8}{10}\), the number of roots Arun harvests, will be less than or equal 240 (see part i.). This means that all of the roots will be sold to individual sellers for 2000 rupees each, for a revenue of \(20 \times \frac{8}{10} \times k = 16k\) hundred rupees.

If \(k > 300\), then \(k \times \frac{8}{10}\), the number of roots Arun harvests, will be greater than 240. Arun makes 4800 in revenue from the first 240 of these harvested roots. Then, any harvested roots beyond the first 240 will be sold to wholesalers. The number of roots sold to wholesalers is given by \(k \times 8/10 - 240\), and as these roots are sold for 1500 rupees each, the total revenue will be \(4800 + 15(k \times 8/10 - 240)\) hundred rupees.

Thus the revenue function is

\[
R(k) = \begin{cases} 
16k & \text{if } k \leq 300 \\
4800 + 15((4/5)k - 240) & \text{if } k > 300
\end{cases}
\]

or

\[
R(k) = \begin{cases} 
16k & \text{if } k \leq 300 \\
12k + 1200 & \text{if } k > 300
\end{cases}
\]
iv) Is your formula for \( R(k) \) continuous? Explain why this makes sense.

**Solution:** Checking both sides of our equation from the previous part when \( k = 300 \), we see that we obtain a value of 4800 on both sides, meaning that the function is continuous, as both pieces are clearly continuous on the rest of their domain. This is in line with expectations as we sell the first 240 harvested roots to the individual sellers regardless of whether we harvest more than that amount, meaning that there are no large jumps in the revenue function at \( k = 300 \).

v) Is your formula for \( R(k) \) differentiable? Explain why this makes sense.

**Solution:** Both functions are linear. However, they have different slopes, meaning that there is a sharp corner at \( k = 300 \). This is expected, because Arun’s selling price changes at \( k = 300 \).

b) Suppose that the cost, in hundreds of rupees, of planting \( k \) cassava roots is

\[
C(k) = \frac{k^3}{75000} - \frac{2k^2}{125} + \frac{92k}{5} + 30.
\]

i) What is the smallest value of \( k \) where profit is positive? You may use a calculator or software to compute this value.

**Solution:** Profit in hundreds of rupees is given by

\[
\pi(k) = R(k) - C(k) = \begin{cases} 
16k - (k^3/75,000 - 2k^2/125 + 92k/5 + 30) & k \leq 300 \\
12k + 1200 - (k^3/75,000 - 2k^2/125 + 92k/5 + 30) & k > 300
\end{cases}
\]

Using a calculator, we see that the function in the first piece is equal to 0 when \( k \approx -11.60, 189.93 \), and 1021.66. The value \( k = 189.93 \) lies in the domain of the first piece and by either looking at the graph or testing the value of \( \pi(k) \) at \( k = 189 \) and 190, we can see that \( \pi \) changes from negative to positive at this point. Since \( k \) is a whole number of roots, \( k = 190 \) is the first value where profit is positive.

ii) For what values of \( k \) does \( MR(k) = MC(k) \) on the interval \([0, 450]\)? You should expect to use the quadratic formula here.

**Solution:** We want to know where \( \pi'(k) = MR(k) - MC(k) = 0 \). Taking the derivative of \( \pi(k) \), we obtain

\[
\pi'(k) = \begin{cases} 
16 - 3k^2/75,000 + 4k/125 - 92/5 & k < 300 \\
12 - 3k^2/75,000 + 4k/125 - 92/5 & k > 300
\end{cases}
\]

The first piece simplifies to \(-3k^2/75,000 + 4k/125 - 12/5\). When we set this equal to zero and use the quadratic formula, we see that \( k = 400 \pm 100\sqrt{10} \approx 83.8 \) and 716.2. The latter solution lies outside the domain of the piece.

The second piece simplifies to \(-3k^2/75,000 + 4k/125 - 32/5\). Setting this equal to zero and using the quadratic formula, we see that \( k = 400 \).

Consequently, we obtain the values of \( k \approx 83.8 \) and \( k = 400 \) where \( MR = MC \).
iii) How many cassava roots should Arun plant? Use calculus to justify your answer.

**Solution:** We want to maximize $\pi(k)$. We have three critical points, $k = 83.8$, 300, and 400, as well as the endpoints at $k = 0$ and 450. Note that because $k$ is a whole number, we should check the values at both 83 and 84. Using our equation for $\pi(k)$ from earlier, we have:

$\pi(0) = -30,$
$\pi(30) \approx \pi(84) \approx -126.6,$
$\pi(300) = 330,$
$\pi(400) \approx 316.7,$ and
$\pi(450) = 315.$

Consequently, he should plant exactly 300 roots to maximize profit.

3. Oil and Water

a) Suppose you are an oil pipeline engineer on a rig in the Gulf of Mexico. The pipeline ruptures, and you begin to see an oil slick forming on the surface of the water. You’d like to determine the rate at which oil is leaking from the pipeline. At the surface you see that the leakage is creating an expanding, circular oil slick that is 3 millimeters thick. From an aerial helicopter you are able to estimate that the oil slick has a radius of 200 meters and is growing at a rate of 4 meters per hour.

i) In the context of this problem, what is the rate you’re interested in? Why might it be hard to directly observe this rate?

**Solution:** We are interested in the rate the oil is leaking, i.e. the change in volume as time goes by, measured in m$^3$/hr. To directly observe this rate would require measuring the flow through the rupture in the pipeline, which might be difficult to access and difficult to measure, given the equipment on the rig.

ii) At the moment you observe the radius and how quickly it is growing, how fast is oil leaking from the pipeline?

**Solution:** Since the volume $V$ of a cylinder is $V = \pi r^2 h$, where $r$ and $h$ are the radius and height, respectively, we substitute in 0.003 for $h$, since the thickness, or height, is 3 millimeters, or 0.003 meters. Differentiating with respect to time, $t$,

$$\frac{d}{dt} V = \frac{d}{dt} \pi r^2 (0.003),$$

$$\frac{dV}{dt} = 2\pi r \frac{dr}{dt} (0.003),$$

and substituting in the quantities 200 for $r$ and 4 for $\frac{dr}{dt}$, we find that the value of $\frac{dV}{dt}$ is

$$\frac{dV}{dt} = 2\pi (200)(4)(0.003) = 4.8\pi \approx 15.0796.$$

Therefore, the oil is leaking at a rate of approximately 15.0796 m$^3$/hr at this moment.
b) You are able repair the pipeline so that no more oil is being added to the slick. You estimate that the surface area of the top of the slick will now remain constant at 100,000 \( m^2 \). The oil slick eventually starts to look long and rectangular as a result of being stretched eastward by the ocean current. Your boat is stationed at the western end of the oil slick, and you observe that the width of the slick is 25 meters and shrinking by 1 meter per hour. Unfortunately, there is a local animal sanctuary just 5000 meters east of the eastern end of the oil slick. This situation is shown in the figure below.

![Diagram of oil slick and sanctuary](image)

i) In the context of this problem, what is the rate you’re interested in? Why are you unable to directly observe this rate?

**Solution:** We are interested in the rate the eastern edge of the oil slick is moving eastward, which we cannot directly observe because the boat is stationed at the western end. Note that this is the same rate at which the southern or northern edges are growing in size.

ii) At the moment you observe the oil slick and how quickly its width is shrinking, how fast is the eastern end of the slick approaching the sanctuary?

**Solution:** Since the area \( A \) of a rectangle \( A = lw \), where \( l \) and \( w \) are the length and width, respectively, we differentiate with respect to time, \( t \),

\[
\frac{d}{dt}A = \frac{d}{dt}lw,
\]

\[
\frac{dA}{dt} = \frac{dl}{dt}w + l\frac{dw}{dt}.
\]

The top of the slick has an area which stays constant, so \( \frac{dA}{dt} = 0 \). Substituting in this value, along with the quantities 25 for \( w \), -1 for \( \frac{dw}{dt} \), and \( \frac{100,000}{25} = 4000 \) for \( l \), the value of \( \frac{dl}{dt} \) is

\[
0 = \frac{dl}{dt}(25) + (4000)(-1),
\]

\[
\frac{dl}{dt} = 160.
\]

Thus the eastern edge of the oil slick is moving at a rate of 160 m/hr at this moment.
iii) Assuming the rate at which the slick stretches eastward remains constant, how long (from the time of your observation) will it take the oil to reach the sanctuary?

**Solution:** If the rate in part (ii) stays constant, since the animal sanctuary is 5000 meters east of the eastern end of the slick, it will take

\[
\frac{5000}{160} = 31.25 \text{ hrs}
\]

for the oil to reach the sanctuary.

c) Your boat contains a water purification tank that pumps in the oil and water mixture and pumps out pure water. The tank consists of a conical “lower hull” and a conical “upper hull” that has its top cut off so that the crew can look into the tank. The upper and lower cones are identical in dimensions (except the top is missing on the “upper hull”). The tank also comes equipped with a indicator light that is green if the liquid is in the lower hull and turns red if the liquid rises into the upper hull. A side profile of the system with dimensions (in meters) is shown here.

![Tank Diagram](image)

i) You look into the tank from above and see that the radius of the circular surface is 2 meters and increasing by 0.1 meters per minute. You also see that the indicator light is green. Is the liquid rising or falling? At what rate is it rising/falling?

**Solution:** Since the liquid lies in the lower hull, an increasing radius indicates that the liquid is rising. To calculate the rate, we note that similar triangles and the side view pictured indicate that, for the radius \(r\) and height \(h\),

\[
\frac{3}{8} = \frac{r}{h}, \text{ or } h = \frac{8}{3}r.
\]

Differentiating with respect to time, \(t\), gives

\[
\frac{dh}{dt} = \frac{8}{3} \frac{dr}{dt}.
\]

Now we use the value \(\frac{dr}{dt} = 0.1\) to get

\[
\frac{dh}{dt} = \frac{8}{3}(0.1) = \frac{8}{30} \approx 0.2667
\]
and the height of the liquid is rising at a rate of approximately 0.2667 m/min.

ii) You look into the tank from above and see that the radius of the circular surface is 1 meter and increasing by 0.3 meters per minute. You also see that the indicator light is red. Is the liquid rising or falling? At what rate is it rising/falling?

**Solution:** Since the indicator light is red and the liquid lies in the cut off upper hull, an increasing radius in this scenario indicates the liquid is falling. There are many ways to approach this; we will consider the just the upper hull (imagine it is not cut off), and let $h$ be the height of the liquid into the upper hull. Now, we have

\[
\frac{3}{8} = \frac{r}{8-h}, \quad \text{or} \quad h = 8 - \frac{8}{3}r.
\]

Again differentiating with respect to time, $t$, gives

\[
\frac{dh}{dt} = -\frac{8}{3} \frac{dr}{dt}.
\]

Using the value $\frac{dr}{dt} = 0.3$, we get

\[
\frac{dh}{dt} = -\frac{8}{3}(0.3) = -\frac{8}{10} = -0.8.
\]

Therefore, the height of the liquid is falling at a rate of 0.8 m/min.