1. Consider the curve $C$ defined implicitly by the equation

$$x(y^2 - 4y) + 8y = 2y^2 - 3x + 5.$$ 

Throughout this problem, be sure to show every step of your algebraic work.

(a) Find all points on the curve $C$ that have a $x$-coordinate of 3.

**Solution:** We plug $x = 3$ into the equation of the curve $C$ and simplify it.

$$3(y^2 - 4y) + 8y = 2y^2 - (3 \times 3) + 5$$

$$\Rightarrow y^2 - 4y + 4 = 0.$$ 

By the quadratic formula or by factoring, it has a double root at $y = 2$. So the only point on the curve $C$ with $x$-coordinate of 3 is the point $(3,2)$.

(b) Find a formula for $dy/dx$. Be sure every step of your work is clear.

**Solution:** Taking the derivative of both sides we obtain

$$(y^2 - 4y) + x(2y - 4)y' + 8y' = 4yy' - 3.$$ 

Then we move each term containing $y'$ to the left side of the equation and the other terms to the right side of the equation to obtain.

$$(2xy - 4x + 8 - 4y)y' = -3 - y^2 + 4y.$$ 

Then dividing in order to isolate $y'$ we find

$$y' = \frac{-y^2 + 4y - 3}{2xy - 4x - 4y + 8}.$$ 

(c) Check that the point $(\frac{5}{3}, 4)$ lies on the curve $C$, and then find the equation of the tangent line to $C$ at this point.

**Solution:** We evaluate both sides at the point $(\frac{5}{3}, 4)$.

$$\text{RHS} = \frac{5}{3} \times \left(4^2 - (4 \times 4)\right) + (8 \times 4)$$

$$= 32,$$ 

$$\text{LHS} = (2 \times 4^2) - (3 \times \frac{5}{3}) + 5$$

$$= 32.$$ 

Since they equal, the point $(\frac{5}{3}, 4)$ lies on the curve $C$. 

Using our answer from the previous part, the slope at this point is
\[
\frac{-4^2 + (4 \times 4) - 3}{2 \times \frac{3}{2} \times 4 - (4 \times \frac{3}{2}) - (4 \times 4) + 8} = \frac{9}{4}.
\]
Thus the tangent line is
\[y = \frac{9}{4} (x - \frac{5}{3}) + 4 = \frac{9}{4}x + \frac{1}{4}.
\]

(d) Find the coordinates of all points where \(C\) has a horizontal tangent line, or explain why there are no such points.

**Solution:** For the points to have a horizontal tangent line, they must have zero slope. So we necessarily have \(-y^2 + 4y - 3 = 0 \Rightarrow y = 1, 3\). But then the equation turns into \(-3x + 8 = 2 - 3x + 5\) and \(-3x + 24 = 18 - 3x + 5\) respectively. Since both have no solutions for \(x\) (the \(-3x\) terms on both sides of each equation cancel), we conclude that there are no such points.

(e) Find the coordinates of all points where \(C\) has a vertical tangent line, or explain why there are no such points.

**Solution:** For a point to have a vertical tangent line, we need the denominator of \(dy/dx\) to equal zero. Thus \(2xy - 4x - 4y + 8 = 0\). This equation is equivalent to
\[x(2y - 4) = 4y - 8.
\]
If \(y\) is not 2, then dividing each side by \(2y - 4\) tells us that \(x = 2\), and if \(y = 2\), both sides of the equation are 0. Thus the denominator of \(dy/dx\) vanishes when \(x = 2\) and \(y = 2\). When \(x = 2\), plugging the coordinate into the equation of the curve, we find
\[2(y^2 - 4y) + 8y = 2y^2 - 6 + 5,
\]
which simplifies to \(0 = -1\), meaning the equation has no solutions. On the other hand, if \(y = 2\), we find
\[x(4 - 8) + 16 = 8 - 3x + 5,
\]
which gives \(x = 3\). Thus the point (3, 2) is the only point on the curve with a vertical tangent line.
2. Isabelle is a bee keeper who wants to sell honey at the local farmers market. Let \( y = H(d) \) be the amount of honey, in pounds, that Isabelle will sell in a month if she charges \( d \) dollars per pound of honey. As Isabelle increases the price per pound, fewer people will buy her honey, so \( H(d) \) is a **decreasing** function. Some values of \( H(d) \) and \( H'(d) \) are given in the table below.

<table>
<thead>
<tr>
<th>( d )</th>
<th>5.25</th>
<th>6.50</th>
<th>7.40</th>
<th>8.10</th>
<th>9.25</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H(d) )</td>
<td>57</td>
<td>48</td>
<td>35</td>
<td>28</td>
<td>22</td>
</tr>
<tr>
<td>( H'(d) )</td>
<td>-7.4</td>
<td>-12.7</td>
<td>-10.2</td>
<td>-7.3</td>
<td>-5.1</td>
</tr>
</tbody>
</table>

Assume that between each pair of consecutive values of \( d \) given in the table, \( H'(d) \) is either always increasing or always decreasing.

(a) Find the linear approximation of \( H(d) \) at \( d = 8.10 \), and use it to estimate the amount of honey Isabelle will sell if she charges $8.00 per pound of honey. Is your estimate an overestimate or an underestimate? Explain.

**Solution:** The tangent line of \( H \) when \( d = 8.10 \) is

\[
l(d) = H'(8.1)(d - 8.1) + H(8.1) = -7.3(d - 8.1) + 28.
\]

Thus

\[H(8) \approx l(8) = -7.3(-0.1) + 28 = 28.73.
\]

Since \( H' \) is increasing on \( 7.40 \leq d \leq 8.10 \), \( H \) is concave up on the interval, meaning the tangent line is below the graph and that the above approximation is an underestimate.

(b) Isabelle’s friend Tom is from France and wants to sell her honey abroad. The amount of honey, in pounds, that Tom will sell if he charges \( k \) Euros per pound of honey is given by \( T(k) = 5\sqrt{H(1.08k)} \). Find the linear approximation of \( T(k) \) at \( k = 7.5 \), and use it to estimate the amount of honey Tom will sell if he charges 8 Euros per pound of honey.\(^1\)

**Solution:** As in part (a), the tangent line of \( T(k) \) will need \( T'(7.5) \) which can be calculated using the chain rule:

\[
T'(7.5) = \frac{5 \cdot 1.08H'(1.08 \cdot 7.5)}{2\sqrt{H(1.08 \cdot 7.5)}} = \frac{2.7H'(8.1)}{\sqrt{H(8.1)}} \approx -3.72.
\]

Thus

\[T(8) \approx T'(7.5) \cdot (8 - 7.5) + T(7.5) \approx -3.72(0.5) + 5 \cdot \sqrt{H(8.1)} \approx 24.59.
\]

(c) Explain why \( H(d) \) is an invertible function.

**Solution:** \( H \) is strictly decreasing, so it must be an invertible function.

(d) Compute \((H^{-1})'(48)\), and give a practical interpretation of your answer.

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\(^1\) As of February 23, 2020, 1 Euro is equivalent to 1.08 dollars. In particular, 7.5 Euros is equivalent to 8.1 dollars.
Solution: We have

\[ (H^{-1})'(48) = \frac{1}{H'(H^{-1}(48))} = \frac{1}{H'(6.5)} = \frac{1}{-12.7} \approx -0.079 \]

which means that when 48 pounds of honey sell per month, in order to increase the amount sold by one pound the price must be decreased by about eight cents.

(e) Suppose Isabelle has exactly 50 pounds of honey. Find the linear approximation of \( H^{-1}(y) \) at \( y = 48 \), and use it to estimate the price Isabelle should charge if she wants to sell all of her honey.

Solution: Following the same procedure as earlier parts, the tangent line at \( y = 48 \) of \( H^{-1}(y) \) is

\[ l(y) = (H^{-1})'(48)(y - 48) + H^{-1}(48) = \frac{1}{-12.7}(y - 48) + 6.5 \]

so

\[ H^{-1}(50) \approx l(50) \approx 6.34 \]

meaning the price should be set to about 6.34 dollars per pound in order to sell 50 pounds.

(f) Let \( R(d) = d \cdot H(d) \) be Isabelle’s revenue in dollars when she charges \( d \) dollars per pound of honey. Compute \( R'(7.4) \) and give a practical interpretation of your answer.

Solution: We have

\[ R'(d) = H(d) + dH'(d) \]

by the product rule. In particular, \( R'(7.4) = 35 + 7.4(-10.2) = -40.48 \) which means when selling honey at a rate of 7.4 dollars per pound, an increase in price by 10 cents per pound will result in a decrease in monthly revenue by about 4.05 dollars.
3. If you deposit $a$ dollars into a bank account with an interest rate of $k\%$ per year, the value of your deposit after $t$ years is given by

$$a \left(1 + \frac{k}{100}\right)^t.$$ 

The Rule of 70 states that the time it takes for an investment like this to double is approximately 70 divided by the percent growth rate. For example, if the value of an investment is growing at a rate of 2% per year, then the Rule of 70 predicts that it will take approximately $\frac{70}{2} = 35$ years for the investment to double in value.

Your work through the first few parts of this problem will justify why the Rule of 70 works. Since the point of the rule is to be a simple approximation that can be done in your head, your work in this problem shouldn’t involve a calculator except perhaps to double-check some basic arithmetic (i.e. addition, subtraction, multiplication, or division).

(a) Show that the number of years it takes for your deposit to double in value is given by

$$\frac{\ln(2)}{\ln \left(1 + \frac{k}{100}\right)} \approx 0.70 \ln \left(1 + \frac{k}{100}\right).$$

**Solution:** Suppose we write $D(t)$ for the value of the deposit after $t$ years. Then $D(t) = a \left(1 + \frac{k}{100}\right)^t$. To find the doubling time $t_d$ for the exponential growth, we solve the equation $D(t_d) = 2a$. That is,

$$\left(1 + \frac{k}{100}\right)^{t_d} = 2.$$ 

Taking the natural logarithm both sides, we obtain $t_d \ln \left(1 + \frac{k}{100}\right) = \ln 2$, which gives

$$t_d = \frac{\ln(2)}{\ln \left(1 + \frac{k}{100}\right)} \approx 0.70 \ln \left(1 + \frac{k}{100}\right). \quad (1)$$

(b) Compute the linear approximation $L(x)$ of $\ln(1 + x)$ at $x = 0$.

**Solution:** The function $f(x) = \ln(1 + x)$ has $f(0) = \ln 1 = 0$, and $f'(0) = \frac{1}{1+0} = 1$ (as $f'(x) = \frac{1}{1+x}$). Therefore,

$$L(x) = f(0) + f'(0)(x - 0) = x.$$ 

(c) Suppose that the interest rate is 8% per year. Use your linear approximation from (b) to estimate $\ln(1.08)$, the denominator of the formula from (a) when $k = 8$. Then, apply this estimate to the formula from (a) to approximate how long it takes for the deposit to double. Compare this calculation to the one described by the Rule of 70.

**Solution:** Using the above linear approximation, we have $\ln(1.08) \approx L(0.08) = 0.08$. With $k = 8$, our formula (1) says that it takes approximately $\frac{0.70}{0.08} = 8.75$ years for the deposit to double. At the same time, the Rule of 70 claims that the time required for the deposit to double is about $70/8 = 8.75$ years, which matches our result.
(d) Now use your linear approximation from (b) to estimate \( \ln \left( 1 + \frac{k}{100} \right) \) for any \( k \) (i.e., without picking any value for \( k \)). How does this justify the Rule of 70?

**Solution:** For general interest rate \( k \), \( \ln \left( 1 + \frac{k}{100} \right) \approx \frac{k}{100} \), hence doubling time \( t_d \approx \frac{0.70}{k/100} = \frac{70}{k} \), which is the formula given by the Rule of 70.

(e) Compute the quadratic approximation \( Q(x) \) of \( \ln(1 + x) \) at \( x = 0 \).

**Solution:** With \( f(x) = \ln(1 + x) \), \( f''(x) = \frac{-1}{(1+x)^2} \) and

\[
Q(x) = f(0) + f'(0)x + \frac{1}{2} f''(0)x^2
= x - \frac{1}{2} x^2.
\]

(f) Using your formula from (a) and your quadratic approximation from (e), estimate the time it will take for your deposit to double in value if the interest rate is 8% per year.

**Solution:** Using an interest rate of \( k = 8\% \), the doubling time is

\[
t_d \approx \frac{0.70}{\ln(1.08)} \approx \frac{0.70}{(0.08 - \frac{1}{2}(0.08)^2)} \approx 9.1145 \text{ years}.
\]