1. Consider the function
\[ f(x) = \begin{cases} 
2x^3 + 3x^2 - 72x + 2 & x < 0 \\
(2x^2 + 3x + 2)e^{-x} & x \geq 0.
\end{cases} \]

(a) What are the critical points of \( f(x) \)? You may need to use the quadratic formula.

\textbf{Solution:} We have
\[ f'(x) = \begin{cases} 
6x^2 + 6x - 72 & x < 0 \\
(-2x^2 + x + 1)e^{-x} & x \geq 0.
\end{cases} \]

We first check where \( f'(x) \) does not exist at \( x = 0 \). This is because by the above formula, the value of \( f'(x) \) approaches \(-72\) as \( x \) approaches 0 from the left, while the derivative approaches 1 as \( x \) approaches 0 from the right, meaning \( f'(x) \) has a corner at \( x = 0 \).

We next check when \( f'(x) = 0 \). Note that \( 6x^2 + 6x - 72 \) factors as \( 6(x+4)(x-3) \). This polynomial has zeroes at \( x = 3 \) and \( x = -4 \), but of these, only \(-4\) is less than 0, meaning that \(-4\) is the only critical point coming from this piece of the function. Next, the function \((-2x^2 + x + 1)e^{-x}\) is equal to zero when \((-2x^2 + x + 1)\) is equal to zero, as \( e^{-x} \) is strictly positive. By the quadratic formula \((-2x^2 + x + 1)\) has zeroes at \( x = 1 \) and \( x = \frac{-1}{2} \), of which only \( x = 1 \) is in the correct piece of the function.

Thus we get \( x = -4, 0, 1 \) as critical points.

(b) Classify each critical point as a local maximum, local minimum, or neither. Use calculus to find and justify your answer.

\textbf{Solution:} We will use the first derivative test. Using our critical points from the previous part, we need to determine the sign of \( f'(x) \) on the intervals \((-\infty, -4), (-4, 0), (0, 1), \) and \((1, \infty)\).

- On the interval \((-\infty, -4)\), we choose the test point \( x = -6 \). Then \( f'(-6) = 108 \), meaning the derivative is positive on this interval.
- On the interval \((-4, 0)\), we choose the test point \( x = -1 \). Then \( f'(-1) = -72 \), meaning the derivative is negative on this interval.
- On the interval \((0, 1)\), we choose the test point \( x = \frac{1}{2} \). Then \( f'(\frac{1}{2}) = e^{-1/2} \), meaning the derivative is positive on this interval.
- On the interval \((1, \infty)\), we choose the test point \( x = 2 \). Then \( f'(2) = -5e^{-2} \), meaning the derivative is negative on this interval.

Thus by the first derivative test, we find that \( x = -4, 1 \) are local maxes and \( x = 0 \) is a local min.

\textbf{Note:} Be sure that your solution for this problem satisfies the standards for justification outlined here.
Solution: We note that the function is continuous at $x = 0$ (both sides approach 2 as $x = 0$), so we evaluate $f(x)$ at the appropriate critical points of each interval and determine the end behavior.

i. $[0, \infty)$

$$f(0) = 2$$

$$f(1) = \frac{7}{e} \approx 2.6$$

$$\lim_{x \to \infty} f(x) = 0$$

The global max is at $x = 1$. There is no global min.

ii. $[-6, 8]$

$$f(-6) = 110$$

$$f(-4) = 210$$

$$f(0) = 2$$

$$f(1) \approx 2.6$$

$$f(8) \approx 0.05$$

The global max is at $x = -4$. The global min is at $x = 8$.

iii. $(-\infty, \infty)$

$$\lim_{x \to -\infty} f(x) = -\infty$$

$$f(-4) = 210$$

$$f(0) = 2$$

$$f(1) \approx 2.6$$

$$\lim_{x \to \infty} f(x) = 0$$

The global max is at $x = -4$. There is no global min.

Note: Be sure that your solution for this problem satisfies the standards for justification outlined here.

(d) What are the inflection points of $f(x)$? Use calculus to find and justify your answer.

Solution: The second derivative is

$$f''(x) = \begin{cases} 
12x + 6 & x < 0 \\
(2x^2 - 5x)e^{-x} & x > 0.
\end{cases}$$
Since \( f'(x) \) is undefined at \( x = 0 \), \( f''(x) \) is undefined there as well.

Next \( f''(x) \) is zero at \( x = -\frac{1}{2} \) and \( x = \frac{5}{2} \).

Thus we need to determine the sign of \( f''(x) \) on the intervals \((-\infty, -\frac{1}{2}), (-\frac{1}{2}, 0), (0, \frac{5}{2}), \) and \((\frac{5}{2}, \infty)\).

- On the interval \((-\infty, -\frac{1}{2})\), we choose the test point \( x = -1 \). Then \( f''(-1) = -6 \), meaning the second derivative is negative.

- On the interval \((-\frac{1}{2}, 0)\), we choose the test point \( x = -\frac{1}{4} \). Then \( f''(-\frac{1}{4}) = 3 \), meaning the second derivative is positive.

- On the interval \((0, \frac{5}{2})\), we choose the test point \( x = 1 \). Then \( f''(1) = -3e^{-1} \), meaning the second derivative is negative.

- On the interval \((\frac{5}{2}, \infty)\), we choose the test point \( x = 3 \). Then \( f''(3) = 3e^{-3} \), meaning the second derivative is positive.

Since the second derivative changes sign at each of the three points, we see that all of \( x = -\frac{1}{2}, 0, \frac{5}{2} \) are inflection points.

**Note:** Be sure that your solution for this problem satisfies the standards for justification outlined here.
2. Let $P(t)$ be the price, in dollars, of a share of Sylph Co.’s stock $t$ hours after 9:30 AM on March 5. Some values of $P(t)$ are given in the table below. Assume that $P(t)$ is differentiable on its whole domain, which includes $[0,6.5]$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>1.2</th>
<th>2.6</th>
<th>4.8</th>
<th>6.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(t)$</td>
<td>34.94</td>
<td>36.73</td>
<td>36.45</td>
<td>35.18</td>
<td>35.75</td>
</tr>
</tbody>
</table>

(a) Luigi has written a program that tells him when to buy or sell shares of Sylph Co. In particular, the program tells him to buy shares when the price of a share is falling at a rate greater than $0.50$ per hour, i.e., when $P'(t) < -0.5$, and it tells him to sell shares when the price of a share is rising at a rate greater than $1.00$ per hour, i.e., when $P'(t) > 1$.

Using this information, for each consecutive pair of times in the table, determine whether Luigi’s program must have told him to buy or sell shares of Sylph Co. at some time in the corresponding interval.

**Solution:** Because the function $P(t)$ is differentiable on its domain, the hypotheses of the mean value theorem are satisfied for every closed interval. Consequently, for each interval, we can compute the average rate of change on that interval. Then, if the average rate of change is greater than 1, then the mean value theorem tells us that the derivative must also be greater than 1 at some point on that interval, in which case the program will tell him to buy shares. Similarly, if the average rate of change is less than $-0.5$, then the program will tell him to sell shares.

- For the interval $[0,1.2]$, the average rate of change of $P(t)$ is
  \[
  \frac{36.73 - 34.94}{1.2 - 0} \approx 1.5.
  \]
  Since the average rate of change is greater than 1, the mean value theorem tells us that the program must have told him to sell shares.

- For the interval $[1.2,2.6]$, the average rate of change of $P(t)$ is
  \[
  \frac{36.45 - 36.73}{2.6 - 1.2} = -0.2.
  \]
  This average rate of change is neither less than $-0.5$ nor greater than 1, so we cannot determine whether the program told him to buy or sell shares.

- For the interval $[2.6,4.8]$, the average rate of change of $P(t)$ is
  \[
  \frac{35.18 - 36.45}{4.8 - 2.6} \approx -0.58.
  \]
  Since the average rate of change is less than $-0.5$, the mean value theorem tells us that the program must have told him to buy shares.

- For the interval $[4.8,6.5]$, the average rate of change of $P(t)$ is
  \[
  \frac{35.75 - 35.18}{6.5 - 4.8} \approx 0.335.
  \]
  The average rate of change is neither less than $-0.5$ nor greater than 1, so we cannot determine whether the program told him to buy or sell shares.
Suppose now that \( P(t) \) has domain \((-\infty, \infty)\) and that \( P(t) \) has exactly two critical points, at \( t = 2.2 \) and \( t = 6.1 \).

(b) Classify each critical point as a local maximum, a local minimum, or neither. If there is not enough information, then explain why.

**Solution:** We will use the first derivative test, which requires us to find the sign of \( P'(t) \) on the intervals \((-\infty, 2.2)\), \((2.2, 6.1)\), and \((6.1, \infty)\).

- On the interval \((-\infty, 2.2)\), we see that \( t = 0 \) and \( t = 1.2 \) both lie in the interval. Since we see that \( P(1.2) > P(0) \) from the table, we may conclude from the mean value theorem (whose hypotheses are satisfied as we remarked in the previous part) that there is some point \( t_0 \) between 0 and 1.2 for which \( P'(t_0) > 0 \), meaning \( P'(t) > 0 \) on the entire interval \((-\infty, 2.2)\).

- On the interval \((2.2, 6.1)\), we see that \( t = 2.6 \) and \( t = 4.8 \) both lie in the interval. Since we see that \( P(2.6) > P(4.8) \) from the table, we may conclude from the mean value theorem that \( P'(t) < 0 \) on the interval \((2.2, 6.1)\).

- We know that the function \( P(t) \) is decreasing on the interval \((2.2, 6.1)\), and we remark that \( P(6.5) > P(4.8) \). Thus \( P(6.5) > P(6.1) \), and so we may conclude from the mean value theorem that \( P'(t) > 0 \) on the interval \((6.1, \infty)\).

Thus at the critical point \( t = 2.2 \), the sign of the derivative \( P'(t) \) changes from positive to negative, meaning we have a local maximum. On the other hand, at \( t = 6.1 \), the sign of the derivative changes from negative to positive, meaning we have a local minimum.

(c) Find the \( t \)-coordinate(s) of all global maximum(s) and global minimum(s) of \( P(t) \) on the interval \([1.2, 4.8]\). If either does not exist or if there is not enough information, explain why.

**Solution:** We see that \( t = 2.2 \) is the only critical point on the interval, and so since a local maximum occurs there, the global maximum occurs there as well.

The global minimum then has to occur at one of the end points. We see directly from the table that \( P(4.8) = 35.18 < 36.73 = P(1.2) \), so the global minimum occurs at \( t = 4.8 \).

(d) Find the \( t \)-coordinate(s) of all global maximum(s) and global minimum(s) of \( P(t) \) on the interval \([0, 6.5]\). If either does not exist or if there is not enough information, explain why.

**Solution:** The global maximum must occur at either an end point or at a local maximum inside of the interval. We see from the table that the global maximum cannot occur at an endpoint, as \( P(1.2) = 36.73 \) is greater than the value at either endpoint. Thus it must occur at a local maximum. Since the only local maximum is at \( t = 2.2 \), this must also be the global maximum.

The global minimum must occur at either an end point or at a local minimum inside of the interval. Since \( P(0) = 34.94 < 35.75 = P(6.5) \), we see that the global minimum must occur at either \( t = 0 \) or the local minimum at \( t = 6.1 \). However, we have no way of comparing these two values, so there is not enough information.
3. Daisy wants to build a new silo on her farm. The sides of the silo will be a cylinder of height \( h \) feet and radius \( r \) feet, and its roof will be a hemisphere of radius \( r \) feet, as shown in the picture below. (She does not need to construct a floor.) Suppose that the material for constructing the sides of the cylinder costs \( \frac{6}{\pi} \) dollars per square foot while the material needed to construct the roof costs \( \frac{20}{\pi} \) dollars per square foot.

Recall that the sides of a cylinder, without the circular ends, have area \( 2\pi rh \). Also note that a sphere has surface area \( 4\pi r^2 \) and volume \( \frac{4}{3}\pi r^3 \).

(a) If Daisy wants to spend 162,000 dollars constructing her silo, find a formula for \( h \) in terms of \( r \).

**Solution:** 162,000 dollars is the total cost of making the sides of the cylinder and making the roof. Hence

\[
162000 = \text{Cost of sides} + \text{Cost of roof} = \frac{6}{\pi} (2\pi rh) + \frac{20}{\pi} \left( \frac{4\pi r^2}{2} \right) = 12rh + 40r^2.
\]

Solving for \( h \) we have \( h = \frac{162000 - 40r^2}{12r} \).

(b) Find a formula for the volume of the silo \( V \), in cubic feet, in terms only of the variable \( r \). Your answer should not include the variable \( h \).

**Solution:**

\[
V = V_{\text{cylinder}} + V_{\text{hemisphere}} = \pi r^2 h + \frac{1}{2} \left( \frac{4}{3}\pi r^3 \right).
\]

Using (a),

\[
V(r) = \pi r^2 \left( \frac{162000 - 40r^2}{12r} \right) + \frac{2}{3} \pi r^3.
\]

(c) Find the domain of the function \( V(r) \).

**Solution:** Since \( r \) and \( h \) are the radius and the height of the cylinder, respectively, they cannot be negative. Note if \( r = 0 \), we do not have a 3-dimensional object to study, while if \( h = 0 \), then the shape would be a hemisphere, which is fine. Thus, we need \( r > 0 \) and \( \frac{162000 - 40r^2}{12r} \geq 0 \). That is, \( r > 0 \) and \( r^2 \leq 162000/40 = 4050 \) (with \( \sqrt{4050} = 45\sqrt{2} \approx 63.640 \)). We conclude the domain of \( V(r) \) is \((0, 45\sqrt{2}]\).

(d) Find the values of \( r \) and \( h \) which maximize the volume of the silo.
Solution: Note that, with the domain in (c), we can rewrite volume as $V(r) = -\frac{8}{3}\pi r^3 + 13500\pi r$. The derivative $V'(r) = -8\pi r^2 + 13500\pi$ is defined for every $r$ on the domain $(0, 45\sqrt{2}]$.

$$V'(r) = 0 \iff r^2 = \frac{13500}{8} \implies r = \sqrt{\frac{13500}{8}} \approx 41.079 \text{ ft.}$$

Then $r = 41.079$ is a critical point of $V(r)$ (on our domain). At this value of $r$, $h \approx 191.703$ feet, and volume is approximately $1.161 \times 10^6$ cubic feet. Since $\lim_{r \to 0^+} V(r) = 0$ and $V(45\sqrt{2}) = 5.398 \times 10^5$, we conclude the volume is maximized when $r = 41.079$ ft and $h = 191.703$ ft.

Note: One can also use first/second derivative test to show $r = 41.079$ is a local maximum. Since it is the only critical point, we can conclude it is the global maximum.

(e) Suppose now that we add the following constraints:

- To ensure that there is sufficient space between the silo and nearby buildings, the diameter of the silo can be at most 110 feet.
- A county ordinance forbids the construction of buildings more than 225 feet high.

How does this change the domain of $V(r)$? Find the values of $r$ and $h$ which maximize the volume of the silo under these constraints.

Hint: Note that the total height of the silo is not $h$.

Solution: On top of the condition $0 < r \leq 63.640$, we have new constraints

$$2r \leq 110 \quad \text{and} \quad h + r \leq 225.$$ 

That is,

$$0 < r \leq 55 \quad \text{and} \quad \frac{162000 - 40r^2}{12r} + r \leq 225. \quad (1)$$

After simplification, the latter becomes $28r^2 + 2700r - 162000 \geq 0$. The quadratic equation $28r^2 + 2700r - 162000 = 0$ has two solutions $41.843$ and $-138.272$, so (1) corresponds to

$$0 < r \leq 55 \quad \text{and} \quad r \geq 41.843.$$ 

Hence, the new domain is $[41.843, 55]$. On this domain, $V(r)$ is continuous and does not have a critical point. Thus the maximum volume occurs at one of the endpoints of the domain. We have

$$V(41.843) \approx 1.1609 \times 10^6, \quad V(55) = 9.388 \times 10^5.$$ 

Therefore, the maximum volume is $1.1609 \times 10^6$ cubic feet, and it occurs when the cylinder has radius 41.843 feet and height 183.158 feet.