Math 115 – Second Midterm Exam

March 27, 2007

Name: SOLUTIONS

Instructor: ______________________ Section Number: ______________

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1. **Do not open this exam until you are told to begin.**
2. This exam has 9 pages including this cover. There are 8 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed two sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to show how you arrived at your solution.
8. Please turn off all cell phones and pagers and remove all headphones.

Problem	Points	Score
1	16	
2	16	
3	12	
4	14	
5	10	
6	12	
7	14	
8	6	
Total	100	

1. (4 points each) For the following statements circle True or False. If the statement is *always* true, explain why it is true. If it is false give an example of when the statement is false. Examples may be formulas or graphs.

   (a) If $y(x)$ is a twice differentiable function, then $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$.

   True

   Consider $y = x^3$: $\dfrac{d^2y}{dx^2} = 6x$, but $\left(\dfrac{dy}{dx}\right)^2 = (3x^2)^2 = 9x^4$.

   False

   (b) There exists a function $f(x)$ such that $f(x) > 0$, $f'(x) < 0$, and $f''(x) > 0$ for all real values of x.

   True

   Consider the function $f(x) = e^{-x}$.

   False

   (c) If h is differentiable for all x and $h'(a) = 0$, then $h(x)$ has a local minimum or local maximum at $x = a$.

   True

   Consider the function $h(x) = x^3$ with $a = 0$.

   False

   (d) If f and g are positive and increasing on an interval I, then f times g is increasing on I.

   True

   $(fg)' = f'g + g'f > 0$
   since f, f', g, and g' are all positive.

   False

2. (4 points each) Suppose f and g are differentiable functions with values given by the table below:

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	2	9	-3	7
3	4	11	15	-19

(a) If $h(x) = f(x)g(x)$, find $h'(3)$.

$$\begin{aligned} h'(x) &= f'(x)g(x) + g'(x)f(x) \\ h'(3) &= f'(3)g(3) + g'(3)f(3) \\ &= (15)(11) + (-19)(4) \\ &= 89 \end{aligned}$$

(b) If $j(x) = \dfrac{(g(x))^3}{f(x)}$, find $j'(1)$.

Using the Quotient Rule:

$$\begin{aligned} j'(1) &= \frac{3(g(1)^2 g'(1)f(1) - (g(1))^3 f'(1)}{(f(1))^2} \\ &= \frac{3(9)^2(7)(2) - (9)^3(-3)}{(2)^2} \\ &= 1397.25 \end{aligned}$$

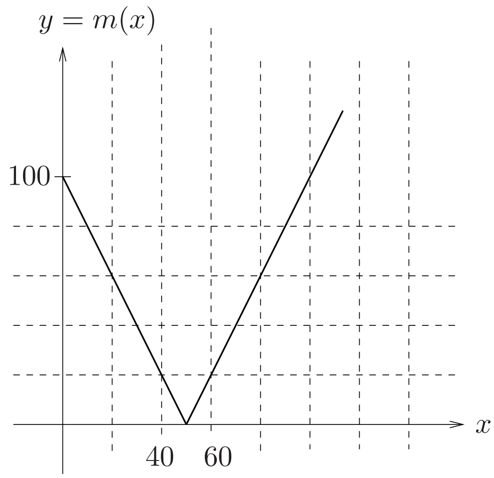
(c) If $d(x) = x \ln\left(e^{f(x)}\right)$, find $d'(3)$.

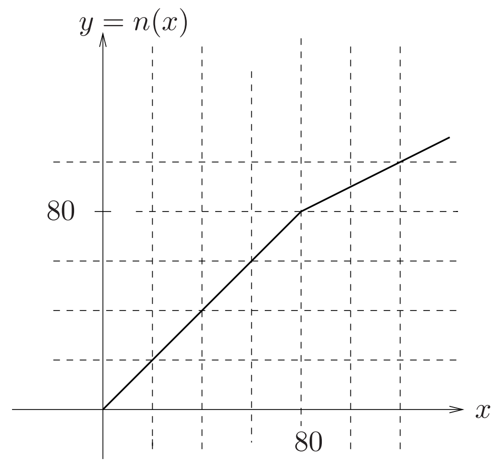
$$\begin{aligned} d(x) &= xf(x) \\ d'(x) &= xf'(x) + (1)f(x) \\ d'(3) &= (3)f'(3) + f(3)(1) \\ &= (3)(15) + 4 \\ &= 49 \end{aligned}$$

(d) If $t(x) = \cos(g(x))$, find $t'(1)$.

$$\begin{aligned} t'(1) &= -\sin(g(1))g'(1) \\ &= -(7)\sin(9) \\ &\approx -2.885 \end{aligned}$$

3. (6 points each) Consider the graphs of $m(x)$ and $n(x)$ below. Let $h(x) = n(m(x))$. Find the following, or explain why they do not exist. The function m has a sharp corner at $x = 50$ and n has a sharp corner at $x = 80$. Determine values that exist as *exact* values–*i.e.*, not a graphical approximation. Please circle your answers.





(a) $h'(80)$

$$\begin{aligned} h(x) &= n(m(x)) \\ \Rightarrow h'(x) &= n'(m(x))m'(x) \\ \Rightarrow h'(80) &= n'(m(80))m'(80) \\ &= n'(60)(2) \\ &= 2 \end{aligned}$$

(b) a value of x such that $h'(x) = -2$

$$h'(x) = n'(m(x))m'(x)$$

We have $m'(x) = -2$ for $0 < x < 50$ and $n'(x) = 1$ for $0 \le x \le 80$.

Thus we need $x = a$ such that $m'(a) = -2$ *and* $0 \le m(a) \le 80$.

Note that any a such that $10 < a < 50$ works.

4. Suppose that x and y satisfy the relation given by the curve

$$x^4 + y^3 = 2 + \frac{7}{2}xy$$

(a) (5 points) Find $\frac{dy}{dx}$.

Differentiating both sides of the above equation with respect to x, we get

$$4x^3 + 3y^2\frac{dy}{dx} = \frac{7}{2}\left(y + x\frac{dy}{dx}\right)$$

Solving for $\frac{dy}{dx}$, we have

$$\frac{dy}{dx} = \frac{\frac{7}{2}y - 4x^3}{3y^2 - \frac{7}{2}x.}$$

(b) (3 points) Under what condition(s) (if any) on x and y is the tangent line to the curve horizontal?

The tangent line to the curve is horizontal where $\dfrac{dy}{dx} = 0$ and the resulting point is on the curve.

The numerator of the derivative is zero when $\frac{7}{2}y - 4x^3 = 0$. Solving for y in terms of x we find that $y = \frac{8}{7}x^3$.

Note that we don't want the numerator to be zero at the same time, so there is a horizontal tangent if $y = \dfrac{8}{7}x^3$ and $x \neq \dfrac{6y^2}{7}$.

(c) (2 points) Consider the points $(1, 2)$ and $(3, 4)$. One of these points lies on the curve, and one does not. Show which point lies on the curve and which does not.

$$\begin{array}{llcl} \text{For (1,2):} & (1)^4 + (2)^3 = 9 & = & 9 = 2 + \frac{7}{2}(1)(2) \\ \text{For (3,4):} & (3)^4 + (4)^3 = 145 & \neq & 44 = 2 + \frac{7}{2}(3)(4) \end{array}$$

Thus (1,2) is on the curve but (3,4) is not.

(d) (4 points) Find an equation of the tangent line to the curve at the point from part (c) that is on the curve.

For the point (1,2), we see that $\frac{dy}{dx} = \frac{6}{17}$. Therefore the equation of the tangent line to the curve at (1,2), using the point-slope formula, is given by

$$y - 2 = \frac{6}{17}(x - 1)$$

Or, in more standard form,

$$y = \frac{6}{17}x - \frac{28}{17}$$

5. (10 points) Find the quadratic polynomial $g(x) = ax^2 + bx + c$ which "best fits" the function $f(x) = \ln(x)$ at $x = 1$ in the sense that

$$g(1) = f(1), \qquad \text{and} \qquad g'(1) = f'(1), \qquad \text{and} \qquad g''(1) = f''(1).$$

$$f(x) = \ln(x) \quad \Rightarrow \quad f(1) = \ln(1) = 0$$

$$f'(x) = \frac{1}{x} \quad \Rightarrow \quad f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2} \quad \Rightarrow \quad f''(1) = -1$$

$$g(x) = ax^2 + bx + c \quad \Rightarrow \quad g(1) = a + b + c$$

$$g'(x) = 2ax + b \quad \Rightarrow \quad g'(1) = 2a + b$$

$$g''(x) = 2a \quad \Rightarrow \quad g''(1) = 2a$$

Thus, using the given equalities, we find

$$2a = g''(1) = f''(1) = -1 \quad \Rightarrow \quad a = -\frac{1}{2}$$

$$2(-\frac{1}{2}) + b = g'(1) = f'(1) = 1 \quad \Rightarrow \quad b = 2$$

$$-\frac{1}{2} + 2 + c = g(1) = f(1) = 0 \quad \Rightarrow \quad c = -\frac{3}{2}$$

$g(x) =$ $-\frac{1}{2}x^2 + 2x - \frac{3}{2}$

6. (12 points) It's time to redesign the layout of exhibits at the San Diego Zoo. The zookeeper, Joan Embery, has been told by Paco Underhill that more exhibits will attract more visitors to enter the zoo but, as the space between exhibits decreases, more zoo visitors are likely to brush butts and flee the zoo in disgust.[1]

   Paco has modeled the predicted number of visitors to the zoo each year, in millions of people, by the function
$$f(x) = axe^{-bx} + c$$
   where x represents the number of exhibits per acre.

   Since the park is beautiful on its own, Paco believes that 1/2 million visitors a year will come to the area, even if there are no exhibits. He has determined that the maximum number of visitors will come to the zoo if there are 4 exhibits per acre. (After that, the "disgust factor" begins to creep in.) According to Paco's model, when the number of exhibits per acre is 4, the number of visitors would be approximately 2.5 million people per year. Use this information (and calculus) to solve for a, b and c in the above function.

   [Note: The maximum number of exhibits that could be packed into an acre of land is 8, since exhibits require 500 m^2, on average, and an acre $\approx$ 4000 m^2.]

   Since half a million visitors will visit the zoo even if there are no exhibits, we have $f(0) = c = \frac{1}{2}$ million visitors. Because the maximum number of visitors occurs when $x = 4$, we know that $f'(4) = 0$:

$$\begin{aligned} f'(x) &= ae^{-bx} - abxe^{-bx}, \\ f'(4) &= ae^{-4b} - 4abe^{-4b} \\ &= ae^{-4b}(1 - 4b). \end{aligned}$$

   Thus, if $f'(4) = 0$, we must have $1 - 4b = 0$, so $b = \dfrac{1}{4}$.

   Because the maximum number of visitors is 2.5 million, we know that $f(4) = 2.5$ (million visitors). Thus

$$2.5 = f(4) = 4ae^{-\frac{1}{4}4} + \frac{1}{2}$$

   Solving for a, we find that $a = \dfrac{e}{2}$.

   [Note: either first or second derivative test should be used to verify that a local maximum DOES occur at $x = 4$–or the endpoints of the interval [0,8] could be used to show that 4 exhibits gives the maximum.]

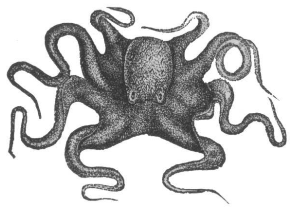
$a =$ $\dfrac{e}{2}$

$b =$ $\dfrac{1}{4}$

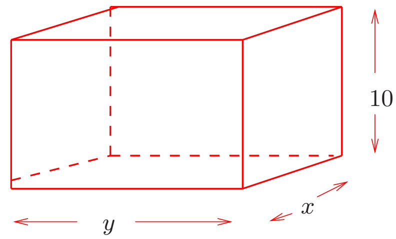
$c =$ $\dfrac{1}{2}$

[1] See http://en.wikipedia.org/wiki/Joan_Embery. and http://www.amazon.com/Why-We-Buy-Science-Shopping/dp/0684849143. When he finishes with the zoo, Underhill will arrange the classroom tables in Dennison Hall.

7. (14 points) No matter what is done with the other exhibits, the octopus tank at the zoo must be rebuilt. (The current tank has safety issues, and there are fears that the giant octopus might escape!) The new tank will be 10 feet high and box-shaped. It will have a front made out of glass. The back, floor, and two sides will be made out of concrete, and there will be no top. The tank must contain at least 1000 cubic feet of water. If concrete walls cost \$2 per square foot and glass costs \$10 per square foot, use calculus to find the dimensions and cost of the least-expensive new tank. [Be sure to show all work.]



GIANT OCTOPUS (*Enteroctopus*)[2]



Denote the width of the tank by x and the length of the tank by y. The height of the tank is given as 10 feet. We know that the volume of the tank must be at least 1000 cubic feet, so let V denote the desired volume of the tank, where $V \geq 1000$. Then for a fixed value of V, we know that $10xy = V$, so that x and y are related by the equation $x = \dfrac{V}{10y}$. Now assuming that one of the y x 10 sides is the front of the tank (i.e, the glass panel), the total cost of the tank is given by:

$$C = 10(10y) + 2[(2)10x + 10y + xy] = 120y + 40x + 2xy.$$

Substituting for x, we can write C as a function of one variable:

$$C(y) = 120y + \frac{4V}{y} + \frac{V}{5}$$

Since the cost function increases as V increases, in order to minimize the cost to build the tank we must have V be as small as possible, so we set $V = 1000$. Our cost equation is now:

$$C = 120y + \frac{4000}{y} + 200$$

Taking the derivative and setting it equal to zero,

$$\frac{dC}{dy} = 120 - \frac{4000}{y^2} = 0,$$

we find that the cost function has a positive critical point at $y = \sqrt{\dfrac{100}{3}} \approx 5.774$ feet. Since the second derivative of the cost function, $\dfrac{d^2C}{dy^2} = \dfrac{24000}{y^3}$, is positive for positive values of y, we know that the function is concave up for all $y > 0$ and this value of y is a minimum for the cost function. Solving for x, we find $x \approx 17.321$ feet.

Thus, the glass side is the small side, and the dimensions and cost are:

Dimensions: 5.774 x 17.321 x 10 feet

Minimum Cost: ~1585.74 dollars

[2]See `http://www.cephbase.utmb.edu/Tcp/pdf/anderson-wood.pdf`. (They really DO escape....)

8. (6 points) On the axes below, sketch a possible graph of a single function $y = g(x)$ satisfying all of the properties below: [Label your points on the axes.]

   (i) $g(x)$ is defined and continuous for all values of x.

   (ii) $g(x)$ has critical points at $x = -1$ and $x = 4$.

   (iii) $g'(x) \geq 0$ on $(-\infty, 4)$.

   (iv) $g(x)$ is decreasing on $(4, \infty)$.

   (v) $\lim_{x\to\infty} g(x) = -2$.