(Solution Guide)

(for students)

Math 115—Exam I

Winter 2001

DEPARTMENT of MATHEMATICS
University of Michigan

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Instructor: ____________ Section No: ____________

General Instructions: *Do not open this exam until you are told to begin.* This test consists of 8 questions on 7 pages (including this cover sheet). The exam is worth 100 points. Do not separate the exam. If any pages do become detached, write your name on them and point them out to your instructor when you turn in the exam.

Please read the instructions for each individual exercise carefully. Show an appropriate amount of work for each exercise so that graders can see not only the answer but also how you obtained it. If you use graphs or tables to obtain an answer, be certain to provide an explanation (and a sketch of the graph, if that is the method) to make it clear how you arrived at your solution. Use units where appropriate.

Problem No.	Points	Score
1	12	
2	5	
3	12	
4	12	
5	10	
6	15	
7	16	
8	18	
Total	100	

1.) (2 pts each) **True / False--Circle your choice. Circle T only if the statement is always true.** [No explanation necessary.]

(a) $\ln(AB) = (\ln A)(\ln B)$ T F

(b) $\ln e^{(2t-1)} = 2t-1$ T F

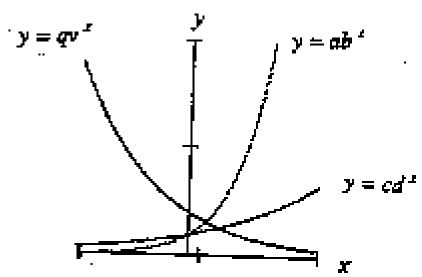
(c) $\sin(3a) = 3\sin(a)$ T F

(d) As $x \to \infty$, x^{100} dominates 1.001^x T F

(e) $\log(10A) = \log A + 1$ $(A > 0)$ T F

(f) A 5th degree polynomial must have at least one real zero. T F

2.) (5 pts--No explanation necessary.) The graphs of three functions are given in the figure below.



Complete each of the statements below by using the symbols $>$, $<$, or $=$.

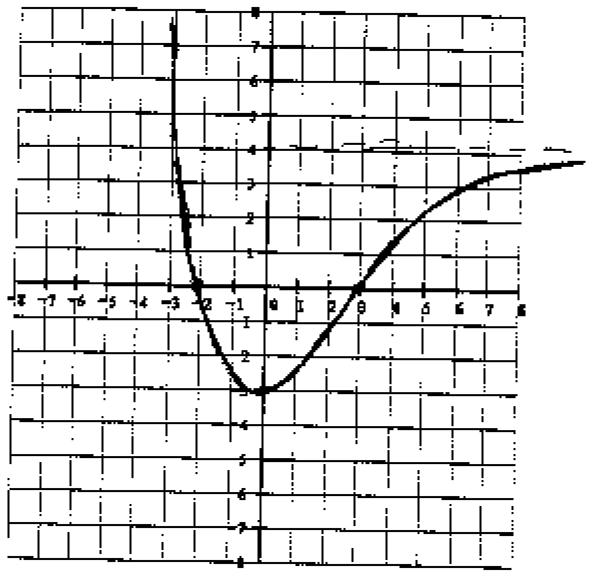
a <u>$<$</u> q a <u>$=$</u> c b <u>$>$</u> d d <u>$>$</u> v

Which, if any, of the parameters a, b, c, d, q, v are greater than zero?

all

3.) (12 pts) (a) On the axes below, sketch a graph of a single *continuous* function, $y = f(x)$, which has *all* of the following features:

- $f(0) = -3$
- $f(-2) = 0$ and $f(3) = 0$
- f is <u>decreasing</u> for $x < 0$
- f is <u>increasing</u> for $x > 0$
- f is <u>concave up</u> for $x < 2$
- f is <u>concave down</u> for $x > 2$
- $f(x) \to 4$ as $x \to \infty$



(b) Is the function you drew in part (a) *invertible*? No

Explain why or why not.

The function does not pass the horizontal line test (or is not one-to-one -- or equivalent...).

4.) Data from three functions is shown in the table below. One function is linear, one is a power function, and one is neither of these.

x	-2	0	2	4	6	8	
$f(x)$	16.5	20	24.2	29.3	35.4	42.9	(neither)
$g(x)$	17.6	20	22.4	24.8	27.2	29.6	✓ – linear
$h(x)$	4.4	0	4.4	17.6	39.6	70.4	✓ Power ...

(a) (6 pts) Determine a formula for the linear function. [Be certain to use the appropriate function name—*i.e.*, f, g, or h, from the table.]

g is linear $m = \frac{22.4-20}{2} = \frac{2.4}{2} = 1.2$

The y-intercept is $b = 20$ so

$g(x) = 20 + 1.2x$

(b) (6pts) Determine a formula for the power function. [Again use the correct function name.]

f can't be a power func. (since $f(0) = 20$)

Try h: $4.4 = k2^p$ & $17.6 = k4^p$, so

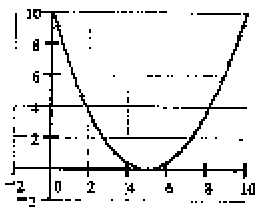
$4 = \frac{17.6}{4.4} = \frac{k4^p}{k2^p} = 2^p$

$h(x) = 1.1x^2$

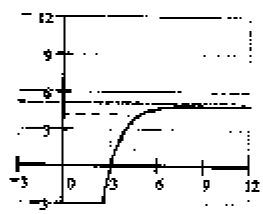
$2^p = 4 \to p = 2$; then $4.4 = k(2)^2$

So, $k = 1.1$

5.) (10 pts) The graphs of f and g are given in the figures below, along with the asymptote to the graph of g.



y = f(x)



y = g(x)

Using the graphs, determine approximate values (to the nearest integer) for each of the following:

(a) $f(g(3))$ 10 $= f(5) =$

(b) $g^{-1}(f(8)) \approx$ 4 $= g^{-1}(4) =$

(c) $f^{-1}(0)$ 5

(d) $f(g(1{,}000{,}000))$ 0 $\approx f(5)$

(e) $g(g^{-1}(3))$ 3

6.) (15 pts) Determine the *zeros* (if any) and describe the behavior as $x \to \infty$ of the following functions: [No explanation necessary.]

(a) $f(x) = \dfrac{5(x+1)(1-x)}{(x+2)(x-3)}$ as $x \to \infty$ $\frac{-5x^2}{x^2} = -5$ zeros: $x = -1,\ x = 1$

as $x \to \infty$, $f(x) \to$ -5

(b) $g(x) = \dfrac{(x^2+1)}{(x+2)}$ as $x \to \infty \simeq \frac{x^2}{x} = x$ zeros: none

as $x \to \infty$, $g(x) \to$ ∞

(c) $h(x) = -2x(x-3)(x+4)$ zeros: $x = 0,\ x = 3,\ x = -4$

cubic with negative leading co-eff.. $\longrightarrow$ as $x \to \infty$, $h(x) \to$ $-\infty$

(d) $j(x) = (x-2)^3(3x+1)$ zeros: $x = 2,\ x = -\frac{1}{3}$

quartic with positive leading co-eff $\rightarrow$ as $x \to \infty$, $j(x) \to$ ∞

(e) Using the function from part (d), write a formula for $m(x)$, given $m(x) = j(x-1)$. [No need to "expand," but do simplify.]

$m(x) = ((x-1)-2)^3(3(x-1)+1)$

$= (x-3)^3(3x-2)$ $m(x) =$ $(x-3)^3(3x-2)$

7.) The populations of Michigan and Arizona between the years of 1960 and 1990 can be modeled by the following functions, where t is the number of years since 1960, and the units of the population is in millions.

Michigan: $f(t) = 7.8(1.0058)^t$; **Arizona:** $g(t) = 1.3(1.035)^t$

(a) (3 pts) [No sentence necessary.] Over the 30 year period, what was the annual percent growth rate for the population of Arizona?

3.5%

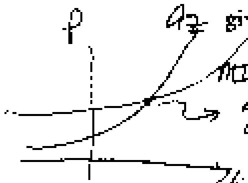
How much greater was that than the corresponding rate for Michigan?

$$\begin{array}{r} 3.50 \\ -\,0.58 \\ \hline \end{array} = 2.92\%$$

(b) (2 pts) What was the difference in the two populations in 1960? [No sentence needed.]

$$\begin{array}{r} 7.8 \\ -\,1.3 \\ \hline \end{array} = 6.5 \text{ million people}$$

(c) (4 pts) If the two states continue to grow according to the patterns given above, will there be a time when the population of Arizona will surpass that of Michigan? If not, explain (mathematically) why not. If so, give the year. [Show your work and express your answer in sentence form.]



$7.8(1.0058)^t = 1.3(1.035)^t \rightarrow \dfrac{7.8}{1.3} = \left(\dfrac{1.035}{1.0058}\right)^t \rightarrow \dfrac{\ln\left(\frac{7.8}{1.3}\right)}{\ln\left(\frac{1.035}{1.0058}\right)} = t \rightarrow t \approx 62.6$

$$\begin{array}{r} 1960 \\ +\,62.6 \\ \hline 2022.6 \end{array}$$

Yes, in the year 2022 the population of Arizona would surpass the population of Michigan.

(d) (2 pts) How many people would the model predict for the population of Michigan in the 2000 census? [No sentence necessary—show work.]

In 2000, $t = 40$, so

$7.8(1.0058)^{40} = 9.83$ million people

(e) (2 pts) Interpret, in the context of this problem, the ***meaning*** of $g^{-1}(2)$. [Sentence form, of course.]

In this model, $g^{-1}(2)$ gives the year that the population of Arizona will have 2 million people.

(f) (3 pts) According to the model above, in what year was the population of Michigan 5 million people? [Show work and express answer in sentence form.]

We want $5 = 7.8(1.0058)^t \rightarrow \dfrac{5}{7.8} = (1.0058)^t$,

so $t \cdot \ln(1.0058) = \ln\left(\frac{5}{7.8}\right) \rightarrow t = \dfrac{\ln\left(\frac{5}{7.8}\right)}{\ln(1.0058)}$

≈ -76.89

$$\begin{array}{r} 1960.00 \\ -\,76.89 \\ \hline 1883.11 \end{array}$$

According to this model, the population of Michigan was 5 million people in the year 1883!

8.) **Essay Question.** *All* **answers should be in complete sentences.**

Average daily temperature for any city in the United States can be approximated with reasonable accuracy by a function of the form $f(t) = A\sin(b(t-h)) + k$, where t is in days after January 1.

For example, a model for average daily temps in the following cities is given by:

Phoenix, AZ:	$f(t) = 20\sin\left(\frac{2\pi}{365}(t-109)\right)+71$
Honolulu, HI:	$f(t) = 4\sin\left(\frac{2\pi}{365}(t-141)\right)+75$
Bismarck, ND:	$f(t) = 30\sin\left(\frac{2\pi}{365}(t-110)\right)+40$

(a) (3 pts) Explain why it is appropriate to use $b = \left(\frac{2\pi}{365}\right)$.

Since $\frac{2\pi}{\text{period}} = b$ & it makes sense for the period to be 365 days, $b = \frac{2\pi}{365}$.

The average temperature in Pittsburgh can be modeled by the function

$$f(t) = 22\sin\left(\frac{2\pi}{365}(t-118)\right)+40\ .$$

(b) (3 pts) According to this model, what is the highest average temperature in Pittsburgh, and in approximately what month during the year does that occur?

The maximum is $40+22=62°$, and this would occur @ $t = \frac{365}{4}+118 = 209.25$ days after Jan 1. Thus, the average temperature is highest in late July in Pittsburgh.

(c) (3 pts) What is the lowest average temperature in Pittsburgh, and in what month does that occur?

The minimum average temperature is $40-22=18°$ and occurs when $t = -\frac{365}{4}+118 = 26.75$ days after Jan 1. Thus, the lowest average temperatures are in January and are ~18°.

[This problem is continued on the next page.]

The model for average daily temperature from the previous page was given as

$$f(t) = A\sin(b(t-h)) + k.$$

(d) (3 pts) In this model, what does the parameter A tell you about the prevailing climate in a city?

The parameter A indicates the maximum number of degrees that the average temperature deviates (above or below) the mean temperature.

(e) (3 pts) What is the effect of the parameter h in the context of these models (*i.e.*, in terms of temperature and days)?

The parameter 'h' has the effect of moving the peak (highest temp) or valley (lowest temp) to different dates in the year.

(f) (3 pts) What does the parameter k indicate in terms of climate?

The constant 'k' indicates the overall yearly average temperature -- or the median between the high & low temperatures.