1. In issue 2 of *Derivative Girl*, our hero has found the Calc1 forgetfulness device, but the code to turn it off involves the values of a function \( f \), and its derivative. To turn off the device, she must enter all the unknown values in the following table:

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( t )</td>
<td>(-2)</td>
<td>(-1)</td>
<td>(0)</td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>( f(t) )</td>
<td>9</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( f'(t) )</td>
<td>2</td>
<td>(-9)</td>
<td>(-1)</td>
<td>(-1)</td>
<td>(-3)</td>
</tr>
</tbody>
</table>

She knows \( f \) is a differentiable function with continuous derivative, and some other facts about \( f \):

- \( \int_{-1}^{0} f'(f(t)) f'(t) \, dt = -7 \),
- \( f(3) = f'(3) \),
- If \( g(z) = z \int_{f(-1)}^{z} f'(t) \, dt \), then \( g'(3) = 5 \),
- \( \int_{-1}^{3} tf''(t) \, dt = -2 \), and
- \( \int_{-1}^{2} f'(t) \, dt = 3 \).

What are the missing values that she will calculate for the table?

**Solution:** We will analyze each piece of information given.

- From \( \int_{-1}^{0} f'(f(t)) f'(t) \, dt = -7 \), we do a substitution with \( w = f(t) \) and obtain

  \[
  \int_{f(-1)}^{0} f'(w) \, dw = -7,
  \]

  and hence \(-7 = f(0) - f(f(-1)) = -f(f(-1))\).

- From \( g(z) = z \int_{f(-1)}^{z} f'(t) \, dt = z(f(z) - f(f(-1))) \), we have that

  \[
  g'(z) = f(z) - f(f(-1)) + zf'(z)
  \]

  Thus

  \[
  5 = g'(3) = f(3) - f(f(-1)) + 3f'(3).
  \]

- From \( \int_{-1}^{3} tf''(t) \, dt = -2 \), we use by parts with \( u = t \) and \( v' = f''(t) \) to obtain

  \[
  -2 = tf'(t)|_{-1}^{3} - \int_{-1}^{3} f'(t) \, dt = (tf'(t) - f(t))|_{-1}^{3} = (3f'(3) - f(3)) - (9 - f(-1)).
  \]
• From $\int_{-1}^{2} f'(t) \, dt = 3$, we directly get that $3 = f(2) - f(-1)$.

Now we summarize all of these.

\[
\begin{align*}
    f(f(-1)) &= 7 \quad (1) \\
    f(3) &= f'(3) \quad (2) \\
    f(3) - f(f(-1)) + 3f'(3) &= 5 \quad (3) \\
    (3f'(3) - f(3)) - (9 - f(-1)) &= -2 \quad (4) \\
    f(2) - f(-1) &= 3. \quad (5)
\end{align*}
\]

Putting (1) and (2) into (3), we get that $4f(3) - 7 = 5$, so $f(3) = f'(3) = 3$.

Putting $f(3) = f'(3) = 3$ into (4), we get that $6 - 9 + f(-1) = -2$, so $f(-1) = 1$.

Finally, putting $f(-1) = 1$ into (5), we have that $f(2) = 4$.

As a summary,

<table>
<thead>
<tr>
<th>$t$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
</tr>
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<tbody>
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<td>$f(t)$</td>
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<td>1</td>
<td>0</td>
<td>7</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>$f'(t)$</td>
<td>2</td>
<td>-9</td>
<td>-1</td>
<td>-1</td>
<td>-3</td>
<td>3</td>
</tr>
</tbody>
</table>

2. Damian’s favorite hobby is convincing his friends that the area of a circle with radius $r$ is $\pi r^2$. He knows this area is given by $2 \int_{-r}^{r} \sqrt{r^2 - x^2} \, dx$. Unfortunately, none of the techniques he tries seems to help in integrating.

(a) His first thought is to try the substitution method with $w = r^2 - x^2$, but this doesn’t work. Do you agree? For the time being, he uses TRAP and MID. Find TRAP(4) and MID(3) when $r = 2$. Is each an overestimate or an underestimate?

Solution: Indeed, substitution with $w = r^2 - x^2$ does not work. For one thing, something strange happens with the limits of integration: we would now be integrating from $w = 0$ to $w = 0$. Additionally, if $w = r^2 - x^2$, then $\frac{dw}{dx} = -2x$, and there isn’t a good way to get this to show up in the integral or to rewrite the integral in terms of $w$ alone after making the substitution.

When $r = 2$, the integral becomes $2 \int_{-2}^{2} \sqrt{4 - x^2} \, dx$.

For TRAP(4), each subinterval is 1 wide (because the total interval has width 4 and we are dividing into 4 equal subintervals) and the Riemann sum is given by

$$2 \cdot 1 \left( \frac{0 + \sqrt{3}}{2} + \frac{\sqrt{3} + 2}{2} + \frac{2 + \sqrt{3}}{2} + \frac{\sqrt{3} + 0}{2} \right) = 4(\sqrt{3} + 1) \approx 10.9282.$$
For MID(3), each subinterval is $\frac{4}{3}$ wide and the Riemann sum is given by

$$2 \cdot \frac{4}{3} \cdot \left(\sqrt{4 - \frac{16}{9}} + 2 + \sqrt{4 - \frac{16}{9}}\right) = \frac{16}{3} \left(1 + \frac{2\sqrt{3}}{3}\right) \approx 13.2838.$$

To decide which is an overestimate and which is an underestimate, we look at the concavity of the integrand $\sqrt{4 - x^2}$. The function is concave down on the entire interval, which tells us that TRAP(4) is an underestimate and MID(3) is an overestimate.

Note that in this case we could also use the fact that we know the area formula of a circle to know that the value of the integral should be $4\pi \approx 12.566$, which confirms that TRAP(4) is an underestimate and MID(3) is an overestimate. (However, it is important that you are able to use concavity to determine this.)

(b) Using sigma notation, write formulas for RIGHT(n) and LEFT(n), again assuming $r = 2$ (for the definition of sigma notation, see page 283 of the text).

**Solution:**

In general, for a given function $f(x)$ on an interval $[a, b]$, RIGHT(n) is given by

$$\sum_{k=1}^{n} f(x_k) \Delta x,$$

where

$$\Delta x = \frac{b - a}{n}$$

is the width of the subintervals and

$$x_0 = a,$$

$$x_n = b,$$

and in general $x_k$ the right endpoint of the $k^{th}$ interval and the left endpoint of the $(k+1)^{th}$ interval. Note that this means that

$$x_1 = a + \Delta x,$$

$$x_2 = a + 2\Delta x,$$

and in general $x_k = a + k\Delta x$.

In this case:

$$f(x) = 2\sqrt{4 - x^2},$$

$$\Delta x = \frac{4}{n},$$

$$x_0 = a = -2,$$

$$x_n = b = 2,$$

and

$$x_k = -2 + k \cdot \frac{4}{n}.$$
So RIGHT(n) is given by
\[
\sum_{k=1}^{n} \left( 2 \sqrt{4 - \left( -2 + \frac{4k}{n} \right)^2} \right) \cdot \frac{4}{n}
\]

The only difference between RIGHT(n) and LEFT(n) is that the height of the rectangle is given by the the left endpoint of each interval instead of the right endpoint. Thus we need our first term to contain \( f(x_0) \) instead of \( f(x_1) \), etc., and the last term will contain \( f(x_{n-1}) \). Thus LEFT(n) is given by
\[
\sum_{k=0}^{n-1} \left( 2 \sqrt{4 - \left( -2 + \frac{4k}{n} \right)^2} \right) \cdot \frac{4}{n}.
\]

Note: we read
\[
\sum_{k=1}^{n} f(x_k) \Delta x,
\]

as “The sum from \( k = 1 \) to \( n \) of \( f \) of \( x \)-sub-\( k \) times delta \( x \).” The idea here is that the \( \Sigma \) tells us that we’re going to add together a list of numbers, each of which has a very similar form, with the only difference that it starts with \( k = 1 \) (ie the number on the bottom of the \( \Sigma \)), and in each term, \( k \) will increase by 1, stopping when \( k \) is equal to the number on top of the \( \Sigma \) (in this case, \( n \)). So
\[
\sum_{k=1}^{n} f(x_k) \Delta x = f(x_1) \Delta x + f(x_2) \Delta x + f(x_3) \Delta x + \cdots + f(x_{n-1}) \Delta x + f(x_n) \Delta x.
\]

Damian’s older brother Richard remembers hearing that sometimes, instead of substituting the more complicated function \( (r^2 - x^2) \) with an easier function \( (w) \), it is helpful to replace the easy function \( x \), with a more complicated function.

For the rest of this problem, let \( r \) be a positive constant (that is, you should no longer assume \( r = 2 \).)

(c) Let’s try this method. Start by letting \( x = r \sin(\theta) \). Now, this makes \( dx = r \cos(\theta) \, d\theta \). Use this replacement to rewrite the integral, including the bounds.

**Solution:** With this substitution, the integral becomes
\[
2 \int_{-\pi/2}^{\pi/2} \sqrt{r^2 - r^2 \sin^2 \theta} \cdot r \cos \theta \, d\theta.
\]

(d) Now, remembering that \( \cos^2(x) + \sin^2(x) = 1 \), simplify the function inside the integral, and finally evaluate it.

**Solution:** Simplifying, we obtain
\[
2 \int_{-\pi/2}^{\pi/2} \sqrt{r^2 - r^2 \sin^2 \theta} \cdot r \cos \theta \, d\theta = 2 \int_{-\pi/2}^{\pi/2} r^2 \cos^2 \theta \, d\theta = 2r^2 \int_{-\pi/2}^{\pi/2} \cos^2 \theta \, d\theta.
\]
This last integral can be evaluated using integration by parts (see Example 6 on Page 356, in Section 7.2 of your textbook). This gives \( 2 \cdot \frac{r^2}{2} \left( \theta + \frac{1}{2} \sin 2\theta \right) \bigg|_{-\pi/2}^{\pi/2} = \pi r^2. \)

Remember that the idea here was to derive the area of a circle, so we can confirm that our integral is correct.

(e) Follow this method with the same replacement, \( x = r \sin(\theta), \) to evaluate \( \int_{-\frac{r}{\sqrt{2}}}^{\frac{r}{\sqrt{2}}} \frac{4r}{\sqrt{r^2 - x^2}} \, dx. \)

Remember \( \sin(\pi/4) = \frac{1}{\sqrt{2}} \) and \( \sin(-\pi/4) = -\frac{1}{\sqrt{2}}. \)

**Solution:** Using the given substitution, we obtain

\[
\int_{-\frac{r}{\sqrt{2}}}^{\frac{r}{\sqrt{2}}} \frac{4r}{\sqrt{r^2 - x^2}} \, dx = \int_{-\pi/4}^{\pi/4} \frac{4r \cdot r \cos \theta}{r \cos \theta} \, d\theta = \int_{-\pi/4}^{\pi/4} 4r \, d\theta = 2\pi r.
\]

Note: If you’d like to learn more about the setup to parts b) through d), check out the textbook §7.4, or look up trig substitution on Khan Academy. This method will not be tested nor required on any course exams (including the gateway), but it gives a way to solve integrals arising in many applications, including ones that show up in Chapter 8.