1. (a) A butterfly lands on a flower, 2 cm above the top of a chamber that has nectar. The chamber has a vertical cross section as shown in the diagram at right, and the horizontal cross sections are ellipse. Let \( \ell(h) \) be the length, in cm, of the narrowest part (the “minor axis”) of the ellipse at a height of \( h \) cm above the bottom of the nectar chamber. Further, the widest part of each ellipse (coming out of the plane in the picture shown), the “major axis”, is double the length of the minor axis.

For each of the following parts, you should give an expression involving one or more integrals to represent the requested quantity. You do not need to compute a value. Be sure to include units.

i. Nectar fills the entire chamber. Give an expression for the total volume of nectar. Note that the area of an ellipse with semi-minor axis length \( a \) and semi-major axis length \( b \) is \( \pi ab \). (The semi-major axis is half of the major axis.)

ii. The density of nectar is given by \( \delta(h) \) mg/cm\(^3\), where \( h \) is height from the bottom of the chamber in centimeters. The chamber of the flower is completely filled with nectar. Give an expression for the total mass of nectar in this chamber of the flower.

iii. Give an expression for the total amount of work the butterfly must do in order to drink all the nectar from this part of the flower.

(b) Nearby, 20 mg of gnats and debris have gotten stuck together in a spider web. The spider, 5 cm above the clump of gnats, pulls the clump up on a strand of its web. The density of the strand of the web is 0.05 mg/cm, and as the spider pulls up the clump, some gnats escape (along with some debris falling off) at a constant rate of 1 mg per centimeter it is pulled up. How much work does the spider have to do to pull the clump of gnats to itself?

2. In issue 4 of Derivative Girl, Darth Integrator has created a very bad rain machine. The machine will cause very bad rain to fall. Anyone caught in very bad rain will have the work they are capable of doing (in joules) cut in half. Not known for his craftsmanship, Darth Integrator’s machine takes some time to warm up. The probability density function \( r(t) \) for the amount of time, in hours, the machine takes to warm up after Darth Integrator plugs it in is given by

\[
r(t) = \frac{a}{1 + (bt)^2} \text{ for } t \geq 0,
\]

where \( a \) and \( b \) are positive constants.
(a) The median amount of time it takes the machine to warm up (after being plugged in) is $1/\pi$ hours. What are the values of $a$ and $b$?

(b) What is the mean amount of time it takes the machine to warm up?

(c) In the spirit of battling Darth Integrator, give an equation for the cumulative distribution function that does NOT involve integrals.

(d) When Darth Integrator plugs in the machine, Derivative Girl is exactly 30 minutes away from it. When she gets there, she will unplug it. What is the probability that she unplugs it before it has finished warming up?

3. You may remember Damian, who loves convincing people the area of a circle is $\pi r^2$. He draws some regular polygons inscribed by a circle with radius $r$. Pictured below are his first few drawings.

(a) He finds out that the area of an inscribed $n-$gon is $nr^2 \frac{\sin \left( \frac{2\pi}{n} \right)}{2}$. Letting $n$ start at 3 for the triangle, he defines

$$ p_n = nr^2 \frac{\sin \left( \frac{2\pi}{n} \right)}{2}. $$

Show this sequence is monotone increasing.

You may with to use the fact that $f(x) = \sin(x) - x \cos(x)$ is positive for $0 < x \leq \frac{2\pi}{3}$ (you do not need to show that this fact is true).

(b) Using the pictures, explain why this sequence is bounded. Putting this together with your answer for (a), what do you know about the convergence of the sequence?

(c) Find the limit of the sequence.

Damian claims your answer for (c) proves the area of a circle is $\pi r^2$. However, his aunt Nyssa says maybe (if she didn’t already know the answer) the area of a circle could be larger than the limit of his sequence. She tells him he will also need the area of a circumscribed regular $n-$gon, given by

$$ P_n = nr^2 \tan \left( \frac{\pi}{n} \right). $$

She shows him the first three, pictured below.

(d) What is the limit of $P_n$?

(e) Use the limits and descriptions of $P_n$ and $p_n$ to briefly explain why the area of a circle must be $\pi r^2$. 
