1. In issue 5 of *Derivative Girl*, Derivative Girl has just stopped Darth Integrator’s very bad rain scheme, but she realizes she can’t leave the machine, as well as the infinite evil horn, out in the open, or else it might fall into the wrong hands. She needs to bring them both to her fouriertress of solitude, to lock them up forever. Remember that Derivative Girl can create a copy of herself (“Copy 1”) that has one-sixth of her strength. Derivative Girl can also create a second copy, but Copy 2 will have \( \frac{1}{6} \) the strength of Copy 1. She can continue doing this, and the \( n \)th copy, “Copy \( n \)”, she creates will have \( \frac{1}{6} \) times the strength of Copy \( (n - 1) \).

(a) Derivative Girl can lift up to 60 tons. The infinite evil horn weighs 71.7 tons. What’s the minimal number of copies she needs to make in order to lift the horn? (She, along with her copies, will team up to lift it.)

**Solution:** Let \( s_i \) be the weight that \( i \)th Copy can lift (in tons), where \( s_0 \) is the weight Derivative Girl herself can lift. Then

\[
\begin{align*}
    s_0 & = 60 \\
    s_1 & = 60 \cdot \frac{1}{6} \\
    s_2 & = (60 \cdot \frac{1}{6}) \cdot \frac{1}{6} = 60 \cdot \left(\frac{1}{6}\right)^2 \\
    s_3 & = \left(60 \cdot \left(\frac{1}{6}\right)^2\right) \cdot \frac{1}{6} = 60 \cdot \left(\frac{1}{6}\right)^3 \\
\end{align*}
\]

In general, \( s_i \) is given by the formula

\[
    s_i = 60 \cdot \left(\frac{1}{6}\right)^i
\]

for all \( i \geq 0 \).

The total weight \( t_n \), in tons, that can be lifted by Derivative Girl and her first \( n \) Copies, is

\[
    t_n = s_0 + s_1 + s_2 + \cdots + s_n
\]

\[
    = \sum_{i=0}^{n} s_i
\]

\[
    = \sum_{i=0}^{n} 60 \cdot \left(\frac{1}{6}\right)^i
\]

This is a finite geometric series with \( n + 1 \) terms, so we can write a formula for the sum in closed form as

\[
    t_n = 60 \cdot \frac{1 - \left(\frac{1}{6}\right)^{n+1}}{1 - \frac{1}{6}}
\]

\[
    = 72 \cdot \left(1 - \left(\frac{1}{6}\right)^{n+1}\right)
\]
We want this quantity to be more than 71.7. We can set up the inequality, and solve for \( n \).

\[
71.7 \leq 72 \cdot \left( 1 - \left( \frac{1}{6} \right)^{n+1} \right)
\]

This inequality simplifies to

\[
\left( \frac{1}{6} \right)^{n+1} \leq 1 - \frac{71.7}{72},
\]

which gives us

\[
n \geq \ln \left( \frac{0.2}{1 - \frac{1}{6}} \right) - 1 > 2.05
\]

Since \( n \) is an integer, that means \( n \) must be at least 3 to satisfy this inequality. Therefore, Derivative Girl needs at least 3 Copies to lift the infinite evil horn.

(b) The very bad rain machine weighs 74 tons. Will they be able to lift it?

**Solution:** Derivative Girl won’t be able to lift the very bad rain machine, since it weighs 74 tons, and no matter how many Copies Derivative Girl makes, they together can lift less than 72 tons, since

\[
\lim_{n \to \infty} t_n = \frac{60}{1 - \frac{1}{6}} = 72
\]

and \( t_n \) is an increasing sequence.

Later in the issue, the copies Derivative Girl makes (Copy 1, Copy 2, Copy 3, etc.) learn how to make their own set of copies, with the same rules. That is, Copy 1 can make a copy, (Copy 1)-Copy 1, that will have \( \frac{1}{6} \) of Copy 1’s strength, and then (Copy 1)-Copy 2, which will have \( \frac{1}{6} \) of (Copy 1)-Copy 1’s strength, and so on, so that her \( n \)th copy, “(Copy 1)-Copy \( n \)”, will have \( \frac{1}{6} \) of (Copy 1)-Copy (\( n - 1 \))’s strength.

(c) What is the heaviest weight that Copy \( m \) along with its own copies can lift?

**Solution:** We know from above that Copy \( m \) can lift \( 60 \cdot \left( \frac{1}{6} \right)^m \) tons.

That means that (Copy \( m \))-Copy 1 can lift \( 60 \cdot \left( \frac{1}{6} \right)^m \cdot \left( \frac{1}{6} \right) \) tons, (Copy \( m \))-Copy 2 can lift \( 60 \cdot \left( \frac{1}{6} \right)^m \cdot \left( \frac{1}{6} \right)^2 \) tons, and in general, (Copy \( m \))-Copy \( k \) can lift \( 60 \cdot \left( \frac{1}{6} \right)^m \cdot \left( \frac{1}{6} \right)^k \) tons.

So together, Copy \( m \) and all its copies can lift

\[
\sum_{k=0}^{\infty} \left( 60 \cdot \left( \frac{1}{6} \right)^m \right) \cdot \left( \frac{1}{6} \right)^k
\]

tons. We can write this in close form as
\[
\frac{60 \cdot \left(\frac{1}{6}\right)^m}{1 - \frac{1}{6}} = 72 \cdot \left(\frac{1}{6}\right)^m
\]

That is, the maximum weight Copy \( m \) along with its Copies can lift is \( 72 \left(\frac{1}{6}\right)^m \).

(d) What’s the heaviest weight that Derivative Girl, the copies, and the copy-copies can lift? Can they lift the very bad rain machine?

**Solution:** Using part (c), we know that all of the copies, together with all of their copies, can lift a total of

\[
\sum_{m=1}^{\infty} 72 \left(\frac{1}{6}\right)^m = 14.4
\]

tons. But this computation does not include Derivative Girl herself, so we must add her strength in as well. This gives us the maximum weight they can all lift together is \( 60 + 14.4 = 74.4 \), so they can lift the very bad rain machine. But only barely—note that adding all these copies has only increased the amount they can lift by 2.2 tons!

Note: The new “copy-copies” made in this process do NOT have the ability to make their own copies. (That is, copying stops after two iterations, there are no copy-copy-copies.) So this really is the largest amount they can lift.
2. Suppose \( a_n, b_n, c_n, \) and \( S_n \) are sequences of positive numbers with the following properties:

\[ \cdot \sum_{n=0}^{\infty} a_{2n} \text{ converges}, \]
\[ \cdot \lim_{n \to \infty} (a_{2n} - a_{2n+1}) = 0, \]
\[ \cdot b_n < a_{2n} \text{ for all } n, \]
\[ \cdot 2 < c_n < 5 \text{ for all } n, \]
\[ \cdot S_n = \sum_{k=0}^{n} b_k \text{ (that is, } S_n \text{ is the partial sum of the first } n \text{ terms of the series } \sum_{k=0}^{\infty} b_k). \]

Determine whether each of the following must converge, must diverge, or whether convergence cannot be determined. Justify your answers.

(a) the sequence \( a_n \)

**Solution:** Because we know \( \sum_{n=0}^{\infty} a_{2n} \) converges, we know that \( \lim_{n \to \infty} a_{2n} = 0 \), that is, the sequence \( a_0, a_2, a_4, \ldots \) converges to zero. By the second bullet point, the sequence \( a_1, a_3, a_5, \ldots \) must also converge to zero, since, when \( n \) is very large, \( a_{2n} \) must be very close to 0, and \( a_{2n+1} \) must be very close to \( a_{2n} \), and therefore also quite close to 0. We conclude that the sequence \( a_n \) must converge to zero.

(b) \( \sum_{n=0}^{\infty} b_n \)

**Solution:** Since \( b_n \leq a_{2n} \) for all \( n \) and \( \sum_{n=0}^{\infty} a_{2n} \) converges, by the (direct) comparison test, the series \( \sum_{n=0}^{\infty} b_n \) must converge.

(c) \( \sum_{n=0}^{\infty} (b_n)^{c_n} \)

**Solution:** We know that the series in (b) must converge, so we must have \( \lim_{n \to \infty} b_n = 0 \). In particular, \( 0 < b_n < 1 \) for large \( n \). Since \( c_n > 2 \), when \( n \) is large enough that \( b_n < 1 \), we must have \( 0 < (b_n)^{c_n} < b_n \). We can therefore use the (direct) comparison test, comparing the series in (c) to the convergent series in (b), and conclude that the series in (c) must converge.

(d) \( \sum_{n=0}^{\infty} \sin(b_n) \) \((Hint: \lim_{x \to 0} \frac{\sin(x)}{x} = 1.)\)

**Solution:** As before, we know that \( \lim_{n \to \infty} b_n = 0 \), so we must have \( 0 < b_n < \pi \) for large enough \( n \). This means that we must also have \( \sin(b_n) > 0 \) for large enough \( n \), so we can apply the limit comparison test, comparing \( \sin(b_n) \) to \( b_n \).
\[
\lim_{n \to \infty} \frac{\sin(b_n)}{b_n} = \lim_{x \to 0} \frac{\sin(x)}{x} = 1
\]

(We can say the first and second limits are equal because \(b_n \to 0\) as \(n \to \infty\) and because \(\sin(x)\) is continuous.)

Since the limit exists and is greater than 0, then our two series either both converge or both diverge. By part (b) we know that \(\sum b_n\) converges, so \(\sum_{n=0}^{\infty} \sin(b_n)\) must converge.

\[\sum_{n=0}^{\infty} \left(\frac{1}{c_n}\right)^n\]

**Solution:** We have \(2 < c_n\), so \(0 < \frac{1}{c_n} < \frac{1}{2}\) and \(0 < \left(\frac{1}{c_n}\right)^n < \left(\frac{1}{2}\right)^n\). Now the series \(\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n\) is a convergent geometric series (because \(|\frac{1}{2}| < 1\), so it follows from the (direct) comparison test that the series \(\sum_{n=0}^{\infty} \left(\frac{1}{c_n}\right)^n\) must converge.

\(\sum S_n\)

**Solution:** The sequence \(S_n\) is the sequence of partial sums for the series \(\sum_{n=0}^{\infty} b_n\). The definition of a series converging is that the partial sums converge. Since we have seen that the series \(\sum_{n=0}^{\infty} b_n\) must converge, the sequence \(S_n\) must converge.

\[\sum_{n=0}^{\infty} S_n\]

**Solution:** Since the terms \(b_n > 0\), the sequence \(S_n = b_0 + b_1 + \cdots + b_n\) is positive and strictly increasing. As a result, \(\lim_{n \to \infty} S_n\) cannot be 0 (the first term must start out greater than 0, and later terms can only move farther away from 0). By the \(n^{th}\) term test for divergence, then, the series \(\sum_{n=0}^{\infty} S_n\) must diverge.

\[\sum_{n=0}^{\infty} a_n\]

**Solution:** In this case, convergence cannot be determined. We can show this by giving two possible examples for \(a_n\), where one example converges, and the other does not. For a convergent example, suppose if \(a_{2n} = a_{2n+1} = 2^{-n}\). This satisfies both bullet points about \(a_n\).
\[ \sum_{n=0}^{\infty} a_{2n} \] is a geometric series with common ratio \( 1/2 \), so it converges.

- \( a_{2n} - a_{2n+1} = 0 \), so certainly the third bullet point is true.

In this example, \( \sum_{n=0}^{\infty} a_n \) is a sum of two convergent geometric series, and therefore converges.

For a divergent example, suppose \( a_{2n} = 2^{-n} \) and \( a_{2n+1} = \frac{1}{n+1} \). The even terms are the same as in the previous example, so certainly \( \sum_{n=0}^{\infty} a_{2n} \) converges. Additionally

\[
\lim_{n \to \infty} a_{2n} - a_{2n+1} = \lim_{n \to \infty} \left( \frac{1}{2^n} - \frac{1}{n + 1} \right) = 0,
\]

so the third bullet point is satisfied. Finally,

\[
\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \left( \frac{1}{n + 1} + 2^{-n} \right).
\]

Since \( \frac{1}{n+1} + 2^{-n} \) and the series \( \sum_{n=0}^{\infty} \frac{1}{n+1} \) diverges (in fact, it is the harmonic series in slight disguise!), we see from the direct comparison test that the series \( \sum_{n=0} a_n \) diverges.
3. A power series is given by \( \sum_{n=0}^{\infty} d_n (x - 10)^n \), where \( d_n > 0 \) for all \( n \).

Suppose additionally we know that this power series converges at 8 and diverges at 16.

(a) Does \( \sum_{n=0}^{\infty} (-1)^n d_n \) converge conditionally, converge absolutely, or diverge, or is there not enough information to decide?

**Solution:**

Note that if we can show that \( \sum_{n=0}^{\infty} |(-1)^n d_n| = \sum_{n=0}^{\infty} d_n \) converges, then we will know that the series we are being asked about is absolutely convergent.

We can get the series \( \sum_{n=0}^{\infty} d_n \) by using the power series with \( x = 11 \), since \( 11 - 10 = 1 \). We are told that the power series converges when \( x = 8 \), so the radius of convergence is at least \(|10 - 8| = 2\). Since \(|11 - 10| < 2\), the power series must also converge at \( x = 11 \).

Therefore the series \( \sum_{n=0}^{\infty} (-1)^n d_n \) is absolutely convergent.

(b) What are all possible values of the radius of convergence?

**Solution:** As in (a), we can see that the radius of convergence is greater or equal to 2. Since \( \sum_{n=0}^{\infty} d_n (x - 10)^n \) diverges for \( x = 16 \) the radius of convergence is at most \( 16 - 10 = 6 \):

\[ 2 \leq R \leq 6. \]

Note that 2 and 6 are both possibilities for the radius of convergence, since the series may converge or diverge at the endpoints of the interval of convergence.

(c) We later find out that \( d_n = \frac{1}{3^n \sqrt{n + e^{-n}}} \). What is the interval of convergence of the power series? Make sure to justify end behavior using appropriate tests.

**Solution:** Let us use the Ratio Test to get a radius of convergence. We get

\[
\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{3^n}{3^{n+1}} \cdot \frac{\sqrt{n + e^{-n}}}{\sqrt{n + 1 + e^{-n-1}}} \cdot \frac{|x - 10|^{n+1}}{|x - 10|^n} \\
= \lim_{n \to \infty} \frac{1}{3} \cdot \frac{\sqrt{n + e^{-n}}}{\sqrt{n + 1 + e^{-n-1}}} \cdot |x - 10| \\
= \frac{1}{3} |x - 10| \cdot \lim_{n \to \infty} \frac{\sqrt{n + e^{-n}}}{\sqrt{n + 1 + e^{-n-1}}}
\]

As \( n \) grows, \( e^{-n} \) approaches 0, so in the long run, the limit above behaves like \( \frac{\sqrt{n}}{\sqrt{n}} = 1 \).

This gives us a limit of \( \frac{1}{3} |x - 10| \). According to the Ratio Test, this converges when \( \frac{1}{3} |x - 10| < 1 \), or \(|x - 10| < 3\). Therefore we have a radius of convergence of 3.

Now, let’s inspect endpoints:
• $x - 10 = 3$. We have a series
\[ \sum_{n=0}^{\infty} \frac{(13 - 10)^n}{3^n \sqrt{n} + e^{-n}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n} + e^{-n}} \]

There are a few ways to show that this series diverges. We cannot use (direct) comparison test with $\frac{1}{\sqrt{n}}$, but we can use limit comparison test.

The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, since it is a $p$ series, with $p = 1/2 \leq 1$. Further,
\[ \lim_{n \to \infty} \frac{\sqrt{n} + e^{-n}}{\sqrt{n}} = \lim_{n \to \infty} \frac{\sqrt{n}}{\sqrt{n} + e^{-n}} = 1, \]
since $n$ dominates $e^{-n}$ as $n \to \infty$. Since this limit is greater than 0, the limit comparison test tells us that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + e^{-n}}$ also diverges. Adding a single term $(n = 0)$ will not change this. Therefore the power series diverges at $x = 13$.

• $x - 10 = -3$. We have an alternating series
\[ \sum_{n=0}^{\infty} \frac{(7 - 10)^n}{3^n \sqrt{n} + e^{-n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n} + e^{-n}} \]

Let $a_n = \frac{1}{\sqrt{n} + e^{-n}}$. Then we are trying to determine the convergence of $\sum (-1)^n a_n$. The terms $a_n = \frac{1}{\sqrt{n} + e^{-n}} \to 0$ as $n \to \infty$ and $0 < a_{n+1} < a_n$. So this series satisfies the conditions of the Alternating Series Test, and therefore converges. In other words, the point $x = 7$ belongs to the interval of convergence.

Summing up, the interval of convergence is $[7, 13)$. 