1. Recall that the density function of a normal distribution with mean $\mu$ and standard deviation $\sigma > 0$ is

$$\frac{1}{\sigma \sqrt{2\pi}} e^{- (x-\mu)^2/(2\sigma^2)}.$$ 

This is an extremely important family of functions in statistics, and models the density function of many naturally occurring phenomena. As you saw in Sections 8.7 and 8.8, we often want to integrate PDFs. The problem here is that there is no elementary formula for an antiderivative of this function. Fortunately, we finally have a method for dealing with this: Taylor series!

Suppose that a large number of students take an exam worth 100 points, and the distribution of student scores on this exam can be modeled by $p(x)$, a normal distribution with mean (and median) 60 and standard deviation 15.

(a) Using the Taylor series for $e^x$ centered at $x = 0$,

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

find the Taylor series for $p(x)$ centered at $x = 60$. Write out both the first four nonzero terms and the entire series using summation notation. (You may find it useful to know that if \( \sum_{n=0}^{\infty} C_n x^n \) is the Taylor series for $f(x)$ centered at $x = 0$, then $\sum_{n=0}^{\infty} C_n (x - a)^n$ is the Taylor series for $f(x-a)$ centered at $x = a$.)

(b) Use your answer to find a Taylor series for the CDF $P(x)$ of this distribution. Again, write out both the first four nonzero terms and the entire series using summation notation. (Warning: think carefully about what the constant term should be!)

(c) Use your formula to confirm that this Taylor series converges for all values of $x$.

(d) Use the degree 5 Taylor polynomial for $P(x)$ to approximate the value of \( \int_{55}^{60} p(x) \, dx \).

Interpret your answer in the context of the problem.

(e) Use the degree 5 Taylor polynomial for $P(x)$ to approximate the percentage of all scores that are within one standard deviation of the mean.

(f) Find $p^{(100)}(60)$, where $p^{(100)}(x)$ is the hundredth derivative of $p(x)$. 
2. In issue 6 of *Derivative Girl*, Darth Integrator sends his minions, The Riemann Sum Brothers, to patrol his garden, directly north of the north wall of his lair. There are three doors on this wall to enter the lair: a back door, a side door, and a front door. The brothers' positions, \((x, y)\), are given by the parametric equations below, where \(x\) is kilometers east of the back door, \(y\) is kilometers north of the lair wall, and \(t \geq 0\) is measured in hours after the brothers start patrolling. The side door is located at \((2, 0)\), and the front door is located at \((4, 0)\).

Left Brother : \[
\begin{align*}
  x &= t^2 \\
  y &= 4 - 2t
\end{align*}
\]

Right Brother : \[
\begin{align*}
  x &= 4 - t \\
  y &= -3(1 - t)^2 + 3
\end{align*}
\]

(a) Darth Integrator is adamant that they start patrolling at 2pm. He tells them exactly what he expects:
- Never be in the same place at the same time,
- Come back into the lair at 4pm for a dastardly meeting, and
- When you do come back in, even if it’s early or late, make sure to use the front door.

Which rules get broken, and by which brother(s)?

Derivative Girl has come down with the Diffy Flu, and has sent in her pet sidekick Gradi-Ant to spy on Darth Integrator. Coincidentally, Gradi-Ant also gets to the garden at 2 pm, with position \((x, y)\) given by:

Gradi-Ant : \[
\begin{align*}
  x &= 4 - \frac{2t}{t+1} \\
  y &= 3e^{\left(\frac{-t}{12}\right)} + 1
\end{align*}
\]

where \(t\) again represents hours after 2 pm.

(b) Who is moving the fastest at 3 pm? Left brother, Right brother, or Gradi-Ant?

(c) What is the length of the path Gradi-Ant walks by the time both brothers are inside the lair?

(d) What direction is Gradi-Ant moving in? (i.e. south, northwest, etc.)

(e) Gradi-Ant’s line of sight is a straight line in the direction Gradi-Ant is moving. What is the \(x\)-coordinate of the spot on the lair wall Gradi-Ant will be looking at at 12 am?

(f) Will the right brother catch Gradi-Ant when he is patrolling? Justify your answer algebraically.

(g) Will the left brother catch Gradi-Ant when he is patrolling? You may use Desmos or a calculator to justify your answer.

(h) Gradi-Ant is very patient, and will spy in the garden for a long time. Darth Integrator is better at finding bugs than the Riemann Sum Brothers, and will catch Gradi-Ant if Gradi-Ant gets closer than 1km north OR east of the back door. Will Darth Integrator catch Gradi-Ant?