1. Integral calculus had been known to Archimedes in the 3rd century BC as the method of exhaustion. Without having the Fundamental Theorem of Calculus at his disposal, Archimedes was still able to calculate area bounded by a parabola. In this problem, we will follow his footsteps to calculate the integral 
\[ \int_0^1 x^2 \, dx \]
without the use of the FTC.
Note that in this problem, we use \( \text{RIGHT}(n) \) (and, respectively, \( \text{LEFT}(n) \)) to denote the right- (and left-)hand Riemann sum with \( n \) equal subintervals.

(a) Approximate the integral \( \int_0^1 x^2 \, dx \) by \( \text{LEFT}(4) \) and \( \text{RIGHT}(5) \). Put the quantities \( \text{LEFT}(4) \), \( \text{RIGHT}(5) \), and \( \int_0^1 x^2 \, dx \) in order from least to greatest (and explain how you know this without computing the value of the integral).

(b) Approximate the integral \( \int_0^1 x^2 \, dx \) by \( \text{RIGHT}(n) \), where \( n \) is a positive integer, and write it in two different ways: one using ellipses and the other using the sigma notation (like the first two parts of Equation (1) below). Note that \( n \) will appear in your answer.

Next consider the following equation, which is true for any positive integer \( n \).
\[ \sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{1}{6} n(n + 1)(2n + 1). \]  
(1)

(c) Verify that Equation (1) is true when \( n = 4 \).

(d) Assuming Equation (1) is true for all positive integers \( n \), rewrite your answer in part (b) into a closed form, that is, without using sigma notation or ellipses (like the last part of Equation (1)).

(e) Find the limit as \( n \) approaches infinity of your answer in (d).

(f) How large could the error between \( \int_0^1 x^2 \, dx \) and \( \text{RIGHT}(n) \) possibly be? (Hint: see “Accuracy of Estimates” at the end of Section 5.1.) What happens as \( n \) approaches infinity?

\footnote{As a fun bonus problem, you might think about why this equation is true. You can find out more about these numbers by looking for information about (square) pyramidal numbers.}
(g) How does your answer to (f) demonstrate that your answer in (e) is the *exact* value of 
\[ \int_0^1 x^2 \, dx \]  
(Remember, your explanation should not involve the Fundamental Theorem of Calculus.)

Almost two thousands years later, the connection between differential calculus and integral calculus was finally recognized. In 1670, the Fundamental Theorem of Calculus was finally proven by Issac Barrow; his idea was later developed by his student, Sir Issac Newton, and systematized by Gottfried Leibniz.

(h) Use the Fundamental Theorem of Calculus instead to compute \[ \int_0^1 x^2 \, dx \]. Is your answer the same as your answer to part (e)?

2. Hannah Haire and Ryan Rabbitt are running a race. They start at the same place at the same time. The graph below shows functions modeling their velocities, in kilometers per hour, \( t \) hours after the race starts. Hannah’s velocity is modeled by \( h(t) \) (solid line) and Ryan’s is modeled by \( r(t) \) (dashed line).

Note that for \( 4 \leq t \leq 5 \) and for \( 5 \leq t \leq 6 \), the graph of \( r(t) \) appears to have the shape of quarter-circles, but these are actually quarter-ellipses due to the scales of the axes. You may use the fact that the area of an ellipse is \( \pi ab \), where \( a \) and \( b \) are the lengths of the semi-major and semi-minor axes. Both graphs are piecewise-linear otherwise.

(a) Compute the total distance traveled by Ryan in the 6 hours.
(b) Calculate Hannah’s average velocity in the 6 hours.
Let $D(t)$ be the distance, in kilometers, that Hannah is *ahead* of Ryan $t$ hours after the race starts. (So $D(t)$ is negative when Hannah is behind Ryan.)

(c) Use the graph to fill out the following table of values$^2$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D(t)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(d) Carefully sketch the graph of $D(t)$. Make sure the following features of the function $D(t)$ are displayed clearly in your sketch:

- the values of $D(t)$ computed in part (c);
- where $D(t)$ is and is not differentiable;
- where $D(t)$ is increasing, decreasing, or constant;
- the concavity of $D(t)$.

3. Suppose that $f$ is a continuous *odd* function, and define another function $F$ by

$$F(x) = \int_{2x-c}^{4} f(t) \, dt,$$

where $c$ is some constant.

(a) Find a value of $c$ for which the graph of $F$ goes through the origin.
(b) Find a value of $c$ for which the graph of $F'$ goes through the origin.

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$^2$Note that Ryan is not capable of teleporting, so his distance from the starting point is a continuous function.