1. (a) Consider a rocket blasting off from the earth’s surface in a video game. At \( t = 10 \) seconds after the rocket engines ignite, the rocket detaches from the platform and lifts off. Its speed after liftoff is given by \( r(t) = \frac{300}{\sqrt{t(t+1)}} \) km/second, where \( t \) is measured in seconds after the engines ignite (i.e. the domain of \( r(t) \) is \([10, \infty)\)). This speed function is accurate for all time after liftoff and the rocket is heading directly away from the earth’s surface.

i. Assuming the video game runs forever, write an expression involving an integral that represents the altitude the rocket approaches as time goes on.

Solution: Recall that distance travelled is given by the integral of speed with respect to time. Since the rocket starts moving at time 10 and goes on forever the appropriate integral is

\[
\int_{10}^{\infty} \frac{300}{\sqrt{t(t+1)}} \, dt.
\]

ii. Find the altitude from part i. Give both an exact value and a decimal value, accurate to at least four digits. Be sure to show and use appropriate notation for your work and calculations. Hint: let \( w = \sqrt{t} \).

Solution: To evaluate this improper integral we introduce a limit:

\[
\lim_{b \to \infty} \int_{10}^{b} \frac{300}{\sqrt{t(t+1)}} \, dt = \int_{10}^{\infty} \frac{300}{\sqrt{t(t+1)}} \, dt.
\]

We proceed by substitution, setting \( w = \sqrt{t} \). In this case \( dw = \frac{1}{2\sqrt{t}} \, dt \), and our integral becomes

\[
\lim_{b \to \infty} \int_{\sqrt{10}}^{\sqrt{b}} \frac{600}{w^2 + 1} \, dw = 600 \lim_{b \to \infty} \int_{\sqrt{10}}^{\sqrt{b}} \frac{1}{w^2 + 1} \, dw.
\]

Noting that \( \arctan(w) \) is an antiderivative of \( \frac{1}{w^2+1} \) we see that this integral is equal to

\[
600 \lim_{b \to \infty} \arctan(w) \bigg|_{\sqrt{10}}^{\sqrt{b}} = 600 \lim_{b \to \infty} \arctan(\sqrt{b}) - 600 \arctan(\sqrt{10}).
\]

Evaluating the limit reveals that

\[
\int_{10}^{\infty} \frac{300}{\sqrt{t(t+1)}} \, dt = 300\pi - 600 \arctan(\sqrt{10}) \text{ km} \approx 183.8 \text{ km}.
\]

(b) During an upcoming untethered space walk, an astronaut in a science fiction movie needs to position herself between 30 and 35 meters away from the space station she is currently aboard. The astronaut’s speed during the walk will be controlled by jets on her spacesuit that operate according to a pre-programmed speed function she cannot change once the walk begins. Like the rocket engines in the previous problem, the jets need 10 seconds to fire up after ignition. Let \( s \) be the number of seconds after the jets first ignite. Note that the speed
functions below are accurate for all time after the jets have fired up, i.e. that the domain of each function is $[10, \infty)$.

i. Will a speed (in meters per second) given by $j_1(s) = \frac{200}{s^2}$ position her within the correct distance of the station, or shoot her off into space? Use appropriate notation and calculations to support your answer.

**Solution:** Again we set up an integral of speed with respect to time to give us distance, namely

$$\int_{10}^{\infty} \frac{200}{s^2} \, ds.$$  

We evaluate this integral just like the last one

$$\int_{10}^{\infty} \frac{200}{s^2} \, ds = \lim_{b \to \infty} \int_{10}^{b} \frac{200}{s^2} \, ds = \lim_{b \to \infty} \left[ 200 \left( -\frac{1}{s} \right) \right]_{10}^{b} = 200 \lim_{b \to \infty} \frac{-1}{b} + 200 \frac{1}{10} = 20.$$  

Since the astronaut travels 20m, which not between 30 and 35, she will not get in the correct position.

ii. After making the previous calculation, she decides that maybe she will instead try to program a speed (in meters per second) of $j_2(s) = \frac{20}{s}$. How far from the space station would she travel with this program?

**Solution:** We do the same thing, noting that her distance travelled is

$$\int_{10}^{\infty} \frac{20}{s} \, ds = \lim_{b \to \infty} \int_{10}^{b} \frac{20}{s} \, ds = \lim_{b \to \infty} 20 \ln(|s|) |_{10}^{b} = 20 \lim_{b \to \infty} \ln(b) - 20 \ln(10).$$

The improper integral diverges since his limit does not exist (is infinite), and we see that with this speed function the astronaut is shot into space, and travels an infinite distance.

iii. After determining these previous two speeds will not work, she considers a speed (in meters per second) of $j_3(s) = \frac{20 \ln 10}{s \ln s}$. What would happen in this case?

**Solution:** In this situation the distance travelled is given by the following integrals:

$$\int_{10}^{\infty} \frac{20 \ln 10}{s \ln s} \, ds = \lim_{b \to \infty} \int_{10}^{b} \frac{20 \ln 10}{s \ln s} \, ds.$$  

Now we use substitution, setting $w = \ln s$, so that $dw = \frac{1}{s} \, ds$, which yields

$$\lim_{b \to \infty} \int_{\ln 10}^{\ln b} \frac{20 \ln 10}{w} \, dw = 20 \ln 10 \lim_{b \to \infty} \ln |w| |_{\ln 10}^{\ln b} = 20 \ln 10 \lim_{b \to \infty} \ln(\ln b) - 20 \ln 10 \ln(\ln 10)$$

This limit does not exist (is infinite); once again the astronaut is shot into the void.
iv. Sketch a graph of the three speed functions from parts i-iii over the interval \([10, \infty)\). What do you notice?

### Solution:

The graphs of all three functions pass through \((10, 2)\). On the interval \((10, \infty)\) \(j_1(s) < j_3(s) < j_2(s)\), with \(\lim_{s \to \infty} j_1(s) = \lim_{s \to \infty} j_2(s) = \lim_{s \to \infty} j_3(s) = 0\).

2. Answer the following using the given table of values:

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>1</th>
<th>(\frac{\pi}{2})</th>
<th>2</th>
<th>3</th>
<th>7</th>
<th>10</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>9</td>
<td>11</td>
<td>15</td>
<td>22</td>
<td>30</td>
</tr>
<tr>
<td>(f'(x))</td>
<td>5</td>
<td>4</td>
<td>1</td>
<td>10</td>
<td>23</td>
<td>12</td>
<td>2</td>
<td>12</td>
</tr>
</tbody>
</table>

The function \(f(x)\) is a positive, differentiable function, and its derivative \(f'(x)\) is continuous and positive. Find the exact value of each of the following.

(a) \(\int_0^{\pi/2} \cos(x)f'(\sin(x))e^{f(\sin(x))} \, dx\)

### Solution:

This problem can be solved using either the Guess-and-Check method observing that

\[
\frac{d}{dx} e^{f(\sin(x))} = \cos(x)f'(\sin(x))e^{f(\sin(x))}
\]

or equivalently using, e.g., the substitution \(w = f(\sin(x))\).

In the first case applying the Fundamental Theorem of Calculus

\[
\int_0^{\pi/2} \cos(x)f'(\sin(x))e^{f(\sin(x))} \, dx = e^{f(\sin(x))}\bigg|_0^{\pi/2} = e^{f(\sin(\pi/2))} - e^{f(\sin(0))} = e^1 - e^0 = e - 1.
\]

Using the substitution \(w = f(\sin(x))\) (so \(dw = \cos(x)f'(\sin(x)) \, dx\)), we obtain

\[
\int_0^{\pi/2} \cos(x)f'(\sin(x))e^{f(\sin(x))} \, dx = \int_{f(\sin(0))}^{f(\sin(\pi/2))} e^w \, dw = e^{f(\sin(\pi/2))} - e^{f(\sin(0))} = e^3 - e^1.
\]
Conclusion: \( \int_0^{\pi/2} \cos(x) f'(\sin(x)) e^{f(\sin(x))} \, dx = e^3 - e. \)

Note that \( e^3 - e \approx 17.3673 \), but the “exact value” (as specified in the instructions) is \( e^3 - e. \)

(b) \( \int_3^7 \frac{15f(x)f'(x) + 25f'(x)}{(f(x) - 3)(f(x) + 2)} \, dx \)

**Solution:** First we use the substitution \( w = f(x) \) (so \( dw = f'(x) \, dx \)) to obtain a rational function:

\[
\int_3^7 \frac{15f(x)f'(x) + 25f'(x)}{(f(x) - 3)(f(x) + 2)} \, dx = \int_{f(3)}^{f(7)} \frac{15w + 25}{(w - 3)(w + 2)} \, dw.
\]

Next, we can apply the method of partial fractions. We need to find constants \( A \) and \( B \) such that

\[
\frac{15w + 25}{(w - 3)(w + 2)} = \frac{A}{w - 3} + \frac{B}{w + 2} = \frac{A(w + 2) + B(w - 3)}{(w - 3)(w + 2)}.
\]

So we must have \( 15w + 25 = A(w + 2) + B(w - 3) \). E.g. by plugging in the special values \( w = -2 \) and \( w = 3 \) we obtain:

\[
-5 = -5B \implies B = 1 \quad 70 = 5A \implies A = 14.
\]

Thus,

\[
\int_{11}^{15} \frac{15w + 25}{(w - 3)(w + 2)} \, dw = \int_{11}^{15} \left( \frac{14}{w - 3} + \frac{1}{w + 2} \right) \, dw = (14 \ln |w - 3| + \ln |w + 2|) \bigg|_{11}^{15} = 14 \ln 12 + \ln 17 - 14 \ln 8 - \ln 13.
\]

Conclusion:

\[
\int_3^7 \frac{15f(x)f'(x) + 25f'(x)}{(f(x) - 3)(f(x) + 2)} \, dx = 14 \ln 12 + \ln 17 - 14 \ln 8 - \ln 13
\]

Note that \( 14 \ln 12 + \ln 17 - 14 \ln 8 - \ln 13 \approx 5.9448 \), but the “exact value” (as specified in the instructions) is \( 14 \ln 12 + \ln 17 - 14 \ln 8 - \ln 13 \) or \( 14 \ln(1.5) + \ln(17/13) \).

(c) \( \int_2^{10} f'(x) \ln(f(x)) \, dx \)

**Solution:** Starting with the substitution \( w = f(x) \) (so \( dw = f'(x) \, dx \)) we obtain

\[
\int_2^{10} f'(x) \ln(f(x)) \, dx = \int_{f(2)}^{f(10)} = 22 \ln(w) \, dw.
\]

Either recalling that

\[
\int \ln(w) \, dw = w \ln w - w + C
\]

or using integration by parts to find

\[
\int \frac{w}{w} \, dw = w \ln w - \int w \, dw = w \ln w - w + C
\]
by the Fundamental Theorem of Calculus, we have
\[
\int_{9}^{22} \ln(w) \, dw = 22 \ln 22 - 22 - (9 \ln 9 - 9) = 22 \ln 22 - 9 \ln 9 - 13.
\]

Alternatively, we could start with an integration by parts, choosing \( u = \ln(f(x)) \) and \( v' = f'(x) \), so that \( u' = f'(x) \) and \( v = f(x) \):
\[
\int_{2}^{10} f'(x) \ln(f(x)) \, dx = \ln(f(x))f(x)|_{2}^{10} - \int_{2}^{10} \frac{f'(x)}{f(x)} f(x) \, dx
\]
\[= (\ln(f(x))f(x) - f(x))|_{2}^{10} = 22 \ln 22 - 22 - (9 \ln 9 - 9).
\]

Conclusion: \( \int_{2}^{10} f'(x) \ln(f(x)) \, dx = 22 \ln 22 - 9 \ln 9 - 13. \)

Note that \( 22 \ln 22 - 9 \ln 9 - 13 \approx 35.2279 \), but the “exact value” (as specified in the instructions) is \( 22 \ln 22 - 9 \ln 9 - 13 \).

3. For all real numbers \( x \), define the function \( G(x) = \int_{4}^{x} 10000 \cos \left( \frac{\pi t^2}{6} \right) \, dt. \)

(a) Calculate \( G'(x) \).

**Solution:** We apply the second fundamental theorem for the function \( f(x) = 10000 \cos \left( \frac{\pi x^2}{6} \right) \)

to see that:
\[
G'(x) = \frac{d}{dx} \int_{4}^{x} 10000 \cos \left( \frac{\pi t^2}{6} \right) \, dt = 10000 \cos \left( \frac{\pi x^2}{6} \right)
\]
for every \( x \).

(b) Write out the terms of a MID(5) estimate of \( G(4.5) \). (You do **not** need to find or approximate the numerical value of your answer.)

**Solution:** We divide the interval \([4, 4.5]\) into 5 equal sub-intervals of length 0.1 with midpoints 4.05, 4.15, 4.25, 4.35, 4.45 respectively. Therefore, we have:
\[
\text{MID}(5) = 0.1 \cdot f(4.05) + 0.1 \cdot f(4.15) + 0.1 \cdot f(4.25) + 0.1 \cdot f(4.35) + 0.1 \cdot f(4.45) =
\]
\[
10000 \cos \left( \frac{\pi (4.05)^2}{6} \right) + 10000 \cos \left( \frac{\pi (4.15)^2}{6} \right) + 10000 \cos \left( \frac{\pi (4.25)^2}{6} \right) +
\]
\[
10000 \cos \left( \frac{\pi (4.35)^2}{6} \right) + 10000 \cos \left( \frac{\pi (4.45)^2}{6} \right).
\]
The last sum is approximately equal to \(-4060.163\).
(c) Determine whether MID\((n)\) and TRAP\((n)\) are overestimates or underestimates of \(G(4.5)\), respectively. Give a formula for, and sketch a graph of a second derivative to explain how you know this. (Feel free to use the help of Desmos or your graphing calculator.)

**Solution:** We need to figure out the concavity of the function \(f(x) = 10000 \cos\left(\frac{\pi}{6} x^2\right)\) in the interval \([4, 4.5]\). By using the chain rule we have that

\[ f'(x) = -\frac{10000\pi}{3} x \sin\left(\frac{\pi}{6} x^2\right) \]

and

\[ f''(x) = -\frac{10000\pi}{3}\left(\sin\left(\frac{\pi}{6} x^2\right) + \frac{\pi}{3} x^2 \cos\left(\frac{\pi}{6} x^2\right)\right). \]

Graphing the function \(f''(x)\) we have the following:

![Graph of f''(x)](image)

We see that indeed \(f''(x)\) is positive on the interval \([4, 4.5]\) and hence \(f(x)\) is concave up on this interval.

Therefore, MID\((n)\) is an **underestimate** for \(G(4.5)\) while TRAP\((n)\) is an **overestimate**.

(d) Consider the estimates \(\text{RIGHT}(100)\), \(\text{LEFT}(100)\), TRAP\((100)\) and MID\((100)\) of \(G(4.5)\). Suppose \(\text{MID}(100) \approx -4027\) and \(\text{LEFT}(100) \approx -4029\).

Recall the relationship between \(\text{LEFT}(n)\), \(\text{RIGHT}(n)\), and TRAP\((n)\). Also recall the relationship between TRAP\((n)\) and MID\((n)\) you established in part (c). Show that \(\text{RIGHT}(100) \geq -4025\).

**Solution:** Recall that \(\text{TRAP}(n) = \frac{1}{2}\left(\text{LEFT}(n) + \text{RIGHT}(n)\right)\). By part (c) we have that \(\text{TRAP}(100) \geq \text{MID}(100)\) and hence \(\text{TRAP}(100) \geq -4027\). Then,

\[ \text{LEFT}(100) + \text{RIGHT}(100) = 2 \cdot \text{TRAP}(100) \geq -2 \cdot 4027 \]

Since \(\text{LEFT}(100) \approx -4029\) we have that

\[ \text{RIGHT}(100) \geq -2 \cdot 4027 - \text{LEFT}(100) \approx -4025 \]