1. Our favorite science fiction actress is studying French in her spare time. She isn’t shooting a movie currently, so she decides she wants to set aside one day a week where she studies the language for a block of at least two hours.

- In the first week, she studies for four hours straight (that is, 2 hours beyond her minimum). However, she’s so burnt out by the length of the study session that she doesn’t study any more that week.

- In the second week, she has a study session for three and a half hours (that is, $\frac{7}{2}$ of an hour, or $\frac{3}{2}$ of an hour beyond her minimum). She finds that she can do another study session later in the second week that’s $\frac{1}{4}$ as long (i.e. $\frac{7}{8}$ of an hour).

- In the third week, she only studies three hours and twenty minutes (that is, $\frac{10}{3}$ of an hour, or $\frac{4}{3}$ of an hour beyond her minimum). She discovers that not only can she do a second study session $\frac{1}{4}$ as long (i.e. $\frac{10}{12}$ of an hour), but also a third study session $\frac{1}{4}$ of the length of the previous session (i.e. $\frac{10}{48}$ of an hour).

(a) If the length of her first study session each week follows this pattern, write down a formula for $s_n$, the length, in hours, of this first study session during week $n$.

**Solution:** We observe that

\[
s_1 = 2 + \frac{2}{1}, \quad s_2 = 2 + \frac{3}{2}, \quad s_3 = 2 + \frac{4}{3}.
\]

so if this pattern continues, the general formula is:

\[
s_n = 2 + \frac{n + 1}{n}.
\]
Assume the actress can keep increasing the number of study sessions by 1 each week as she decreases the duration of the first study session. Let $t_n$, be the total time the actress studies French (in hours) during week $n$.

(b) Find an expression for each of $t_1$, $t_2$, $t_3$, and $t_4$ in terms of $s_1$, $s_2$, $s_3$, and $s_4$, respectively.

**Solution:** Possible answers include:

\[
t_1 = s_1
\]
\[
t_2 = s_2 + \frac{1}{4}s_2 = s_2 \frac{1 - \left(\frac{1}{4}\right)^2}{1 - \frac{1}{4}} = s_2 \frac{5}{4}.
\]
\[
t_3 = s_3 + \frac{1}{4}s_3 + \left(\frac{1}{4}\right)^2 s_3 = s_3 \frac{1 - \left(\frac{1}{4}\right)^3}{1 - \frac{1}{4}} = s_3 \frac{21}{16}.
\]
\[
t_4 = s_4 + \frac{1}{4}s_4 + \left(\frac{1}{4}\right)^2 s_4 + \left(\frac{1}{4}\right)^3 s_4 = s_4 \frac{1 - \left(\frac{1}{4}\right)^4}{1 - \frac{1}{4}} = s_4 \frac{85}{64}.
\]

(c) Write down a formula for $t_n$ in terms of $n$. Your formula should not include the letter “$s$”. This formula should be in closed form; in other words, no summation signs, ellipses, or recursion should appear in your formula.

**Solution:** Note that $t_n$ is the sum of a finite geometric series. In particular, we have

\[
t_n = s_n + \frac{1}{4}s_n + \left(\frac{1}{4}\right)^2 s_n + \left(\frac{1}{4}\right)^3 s_n + \ldots + \left(\frac{1}{4}\right)^{n-1} s_n
\]
\[
= \sum_{i=0}^{n-1} s_n \left(\frac{1}{4}\right)^i
\]
\[
= s_n \frac{1 - \left(\frac{1}{4}\right)^n}{1 - \frac{1}{4}}
\]
\[
= \left(2 + \frac{n+1}{n}\right) \frac{1 - \left(\frac{1}{4}\right)^n}{1 - \frac{1}{4}}
\]

(d) As the weeks go by, does the actress’s study time approach a certain value? If so, what is this value?

**Solution:** We have

\[
\lim_{n \to \infty} t_n = 3 \frac{1}{1 - \frac{1}{4}} = 4 \text{ hours.}
\]

We conclude that, as the weeks go by, the actress’s study time approaches four hours.
2. Determine whether the following integrals converge or diverge. You do not (necessarily) need to calculate the values of the integrals if they converge. Show all steps with proper notation and justify your work.

(a) \( \int_{10}^{\infty} \frac{1}{x \ln\left(\frac{1}{2} x - 1\right)} \, dx \)

**Solution:** Method 1: Note that since \( f(x) = \ln(x) \) is an increasing function, the following inequality holds in \([10, \infty)\).

\[
\frac{1}{x \ln\left(\frac{1}{2} x - 1\right)} \geq \frac{1}{x \ln(x)}
\]

By the comparison test, it will suffice to show that \( \int_{10}^{\infty} \frac{1}{x \ln(x)} \, dx \) diverges.

\[
\int_{10}^{\infty} \frac{1}{x \ln(x)} \, dx = \lim_{b \to \infty} \int_{10}^{b} \frac{1}{x \ln(x)} \, dx
\]

\[
= \lim_{b \to \infty} \int_{\ln(10)}^{\ln(b)} \frac{1}{w} \, dw \quad \text{(using the substitution } w = \ln(x)\text{)}
\]

\[
= \int_{\ln(10)}^{\infty} \frac{1}{w} \, dw \quad \text{which diverges by the } p \text{-test for } p = 1.
\]

Method 2: Alternatively, one may first perform the substitution \( w = \frac{1}{2} x - 1 \). This substitution turns the given improper integral into \( 2 \int_{\frac{1}{2}}^{\infty} \frac{1}{(2w + 2) \ln(w)} \, dw \).

At this point, one needs to find a “smaller function” whose integral over the same interval diverges. Note that \( \frac{1}{2w \ln(w)} \) is actually a bigger function, so showing that the integral of \( \frac{1}{2w \ln(w)} \) diverges is not enough. On the other hand, for \( w \geq 4 \), we have

\[0 \leq \frac{1}{(2w + 2) \ln(w)} \leq \frac{1}{3w \ln(w)} \]

So \( \frac{1}{3w \ln(w)} \) is a smaller function whose integral diverges (as can be computed directly as in Method 1), so applying the Comparison Test will work as well. (One must use an appropriate comparison function. Comparing with \( \frac{1}{3w \ln(w)} \) works, but comparing with \( \frac{1}{2w \ln(w)} \) does NOT.)

Either one of these methods shows that the given integral diverges.

(b) \( \int_{0}^{1} \frac{1}{5 + \sqrt{x}} \, dx \)

**Solution:** Method 1: The integrand is continuous on the interval \([0, 1]\), so this is not an improper integral. That is, this definite integral certainly equals a real number value, i.e. it converges.

Method 2: Note that the following inequality holds on \((0, 1)\): \(0 \leq \frac{1}{5 + \sqrt{x}} \leq \frac{1}{\sqrt{x}}\).

The integral \( \int_{0}^{1} \frac{1}{\sqrt{x}} \, dx \) converges by the \( p \)-test for \( p = 0.5 \) over the interval \((0, 1)\). By the comparison test, that means that the integral \( \int_{0}^{1} \frac{1}{5 + \sqrt{x}} \, dx \) must also converge.
(c) \[ \int_0^1 \frac{8}{x^3(4 + \cos x)} \, dx \]

**Solution:** The following inequalities hold for all \( x \)-values, and in particular on \((0, 1)\).

\[-1 \leq \cos x \leq 1 \quad \text{so} \quad 3 \leq 4 + \cos x \leq 5.\]

In particular, this gives us the following inequality for our integrand.

\[ \frac{8}{x^3(4 + \cos x)} \geq \frac{8}{5x^3} \]

The integral \( \int_0^1 \frac{8}{5x^3} \, dx \) diverges by the \( p \)-test for \( p = 3 \) over the interval \((0, 1)\), which means \( \int_0^1 \frac{8}{x^3(4 + \cos x)} \, dx \) also diverges by the comparison test.

(d) \[ \int_0^1 -\frac{\ln x}{5 - \cos x} \, dx \]

**Solution:** Note that \(-\ln(x) \geq 0\) on the interval \((0, 1]\). Using the fact that \(-1 \leq \cos x \leq 1\) (and therefore \(4 \leq 5 - \cos x \leq 6\)), we get the following inequality for our integrand on the interval \((0, 1]\).

\[ 0 \leq -\frac{\ln x}{5 - \cos(x)} \leq -\frac{\ln x}{4} \]

It will therefore suffice to show that \( \int_0^1 -\ln x \, dx \) converges (which implies \( \int_0^1 -\frac{\ln x}{4} \, dx \) converges): that will mean our integral converges by the comparison test. Recall that one can find antiderivatives of \( \ln x \) using integration by parts.

\[
\int_0^1 \ln x \, dx = \lim_{b \to 0^+} \int_b^1 \ln x \, dx \\
= \lim_{b \to 0^+} (x \ln x - x)|_b^1 \\
= \lim_{b \to 0^+} \left( (-1) - (b \ln b - b) \right) \\
= -1 + \lim_{b \to 0^+} \left( \frac{-\ln b}{1/b} \right) + \lim_{b \to 0^+} (b) \\
= -1 + \lim_{b \to 0^+} \left( \frac{-1/b}{-1/b^2} \right) + 0 \quad \text{(By L’Hôpital’s rule)} \\
= -1 + \lim_{b \to 0^+} b \\
= -1
\]

Since \( 0 \leq -\frac{\ln x}{5 - \cos(x)} \leq -\frac{\ln x}{4} \) on \((0, 1]\) and \( \int_0^1 -\frac{\ln x}{4} \, dx \) converges, the given integral \( \int_0^1 -\frac{\ln x}{5 - \cos x} \, dx \) also converges by the Comparison Test.
3. Note that the divergence/convergence of the series in parts (a) and (b) below follows from the $p$-test for series (or the Integral Test). However, in this problem, you will use more direct approaches to see why the series behave as they do.

(a) Consider the infinite series

\[ \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \]

When first encountering this infinite series (called the “harmonic series”), it is often difficult to believe that

\[ \lim_{n \to \infty} \frac{1}{n} = 0 \quad \text{even though} \quad \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges.} \]

What follows is an argument for the divergence of \( \sum_{n=1}^{\infty} \frac{1}{n} \).

- Ignore the first term, 1, for the moment (it doesn’t quite fit the following pattern).
- Now group the rest of the terms as follows:
  - the first set (in red) contains (only) \( \frac{1}{2} \),
  - the second set (in blue) contains \( \frac{1}{3} \) and \( \frac{1}{4} \),
  - the third set (in orange) contains \( \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \) and \( \frac{1}{8} \).

This is illustrated as:

\[ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \]

i. In general, how many terms are in the \( n \)th set?

**Solution:** The \( n \)th set contains \( 2^{n-1} \) terms.

ii. Each term (the only term) in the first set is greater than or equal to \( \frac{1}{2} \). Each term in the second set is greater than or equal to \( \frac{1}{4} \). Each term in the third set is greater than or equal to \( \frac{1}{8} \). In general, each term in the \( n \)th set is greater than or equal to what value?

**Solution:** Each term in the \( n \)th set is greater than or equal to \( \frac{1}{2^{n}} \).

iii. Combining the previous two parts, the sum of the terms in each set adds up to at least what value?

Write a couple of sentences about how this implies the infinite series \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges.

**Solution:** Since the \( n \)th set contains \( 2^{n-1} \) terms, and each of these terms is greater than or equal to \( \frac{1}{2^{n}} \), the sum of the terms in the \( n \)th set adds up to at least \( 2^{n-1} \cdot \frac{1}{2^{n}} = \frac{1}{2} \).

This implies that sum of all the terms in the first \( n \) sets is greater than \( \frac{n}{2} \).

Stated more precisely, since there are

\[ 1 + 2 + 4 + 8 + \cdots + 2^{n-1} = \sum_{k=1}^{n} 2^{k-1} = \frac{1 - 2^{n}}{1 - 2} = 2^{n} - 1 \]
terms in the first \( n \) sets, we see that (including the first term 1) for any integer \( m \geq 2^n \), the partial sum \( S_m > \frac{n}{2} \).
This implies that the infinite series diverges.

(b) On the other hand, consider the infinite series

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots
\]

For this infinite series,

\[
\lim_{n \to \infty} \frac{1}{n^2} = 0 \quad \text{as well, but} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges.}
\]

Here’s an argument for the convergence of \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) using the picture above.

i. In the infinite series below, indicate how the color-coded sets of terms correspond to the color-coded squares in the picture above.

\[
1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} + \frac{1}{11^2} + \frac{1}{12^2} + \frac{1}{13^2} + \frac{1}{14^2} + \frac{1}{15^2} + \cdots
\]

**Solution:** The terms in the infinite series correspond to areas of the squares in the picture above.

ii. Explain why none of the stacks of squares (even if we add more stacks of squares to the picture corresponding to more sets of terms) have a combined height greater than 1.

**Solution:** Including the first term 1 this time, calling the first stack of squares the large 1x1 square in orange, the second stack the two blue squares \( \left( \frac{1}{2^2} \text{ and } \frac{1}{3^2} \right) \), the third stack the four red squares \( \left( \frac{1}{4^2}, \frac{1}{5^2}, \frac{1}{6^2}, \text{ and } \frac{1}{7^2} \right) \), etc., notice that there are \( 2^{n-1} \) squares in the \( n \)th stack and the greatest height of any of the squares within that stack is \( \frac{1}{2^{n-1}} \).

Consequently, the sum of the heights in the \( n \)th stack is at most \( 2^{n-1} \cdot \frac{1}{2^{n-1}} = 1. \)
iii. Explain, using geometric series, why even as we add more stacks corresponding to more sets of terms, the overall width never exceeds 2.

**Solution:** As in the previous part, the greatest width within each stack (the width of the bottom square in the stack) is \( \frac{1}{2^{n-1}} \). Consequently, adding more and more stacks results in an overall width corresponding to the geometric series

\[
\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-1/2} = 2.
\]

iv. Explain how this picture implies that the infinite series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges.

**Solution:** As noted in part i, the infinite series corresponds to the areas of the squares in the picture. As these areas lie within a rectangle of area 2, the sum of the areas of the squares must be less than 2. That is, the infinite series must converge (to a value less than 2).