The Checkerboard’s Price

Perusing through the Ann Arbor Art Fair, as Ali and Elie were seeking a special gift for their friend Lee, they came across an amazing object: an infinitely long and wide checkerboard. The merchant explained to them that each square on the board is assigned a value, and the price of the whole checkerboard is the sum of all those values (see picture below). He also explained that nobody would buy the gift, because most people think its price is infinite. But Ali and Elie have taken Calculus, and they know the sum of infinitely many terms may be finite. Actually, they each came up with a method to calculate the checkerboard’s price.

In this problem, $0 < x < 1$.

**Ali’s Method:** Ali first added up the squares in the first row, and found its value to be $\frac{x}{1-x}$. For the 2nd row, he found the value $\frac{x^2}{1-x}$; and for the 3rd row, the value was $\frac{x^3}{1-x}$.

(1) Continue what Ali started to find the value of the $n^{th}$ row. **Explain your answer.**

(2) Use your answer to (1) to deduce the price of the checkerboard.

**Elie’s Method:** Elie divided the board diagonally as follows: the first diagonal has 1 square of value $x$, the second diagonal has 2 squares of value $x^2$, the third diagonal has 3 squares of value $x^3$, etc...

(3) Adding altogether the price of each diagonal, show that the board costs $\sum_{n=1}^{\infty} n x^n$.

(4) Find the value to which the series in (3) converges by differentiating with respect to $x$ the formula $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ and modifying the yield appropriately.

(5) Do Ali’s and Elie’s results agree?