The Pythagorean Theorem tells us how to compute the length of the line segment between two points in the Cartesian plane, and this can be generalized to three-space and beyond. But, what if we want to calculate the length of a path that is not a line segment? For example, how to calculate the length of the path that a plane follows as it travels from Ann Arbor to Augusta, Australia? You will learn how to do this in Math 215.

Similarly, you have learned formulas for the area of triangles, squares, and other bounded planar regions. But, what if you wanted to calculate the surface area of an intake manifold, the top of U(M)’s solar car, or the hull of U(M)’s concrete canoe? You will learn how to do this in Math 215.

In calculus you learned various algorithms, e.g., the first derivative test, for optimizing functions of one variable. Optimizing functions of many variables is a very rich subject with important applications in economics, statistics, data science, machine learning, and most every field of science and engineering. For example, an airline needs to schedule flights. How many flights should occur between cities each day? When should they be scheduled? The answers to these and similar questions are constrained by the number and location of aircraft, availability of flight crews, connections, maintenance, passenger demand, profit margins … In Math 215 we will introduce some of the basic tools of this vast and active field of study.

Finally, in calculus you learned that if $F: \mathbb{R} \to \mathbb{R}$ is an antiderivative of $f: \mathbb{R} \to \mathbb{R}$, then
\[ \int_a^b f(x) \, dx = F(b) - F(a). \]

Why isn’t the answer $F(a) - F(b)$? The answer to this question is: orientation. On $\mathbb{R}$, orientation is something we take for granted – we move “left to right” or from negative numbers to positive numbers. However, for curves or surfaces in higher dimensions it is not clear how we should choose which direction to move or which direction to call “up”. For example, for a circle in the plane we can move in one of two directions: clockwise or counterclockwise. Which direction do we choose? You will learn how to do this in a consistent manner in Math 215.

The five homework exercises below explore/review the ideas introduced above.

1. Prove the Pythagorean theorem.

2. (a) What is the area of a right triangle with sides of length $a \leq b \leq c$?
   
   (b) Suppose a parallelogram has sides of length $a$ and $b$. Is this enough information to compute the area of the parallelogram? If not, what additional information is required?
   
   (c) Suppose $C$ is a circle of radius $R$. What is the length of the arc on $C$ that is subtended by a central angle of $\theta$ radians?

3. The demand function for bubble tea is $d(c) = 7 - .0015c$, where $d$ is measured in dollars and $c$ is the number of cups of tea produced and sold. The total cost, in dollars, of producing $c$ cups of bubble tea is $t(c) = 40 + 1.25c$. Determine the level of production that maximizes profit.

---

1. Drawing a right triangle with side lengths 3, 4, 5 and then writing “$3^2 + 4^2 = 5^2$” is not a proof. Your proof should use high school geometry, and it should work for an arbitrary right triangle in the plane. We do not expect you to produce an original proof. Rather, we expect you to find (for example, you may wish to do a bit of research in a library) a solution, understand the solution, and reproduce the solution. Please properly acknowledge all references you use. Also note that a proof which uses $\sin$ or $\cos$ probably is not a proof. Also, handing in a copy of something you find on the Internet is not acceptable.

2. As in the movie Merry Andrew.

3. Profit is described by the equation $cd(c) - t(c)$. Do you see why?
4. The surveyor’s formula (also called the Shoelace formula or Gauss’s area formula) is a handy tool for computing the area of polygonal regions in the plane. For a triangle, it says the following: Suppose our triangle has vertices \((a, b)\), \((c, d)\), and \((e, f)\) oriented in a counterclockwise manner as in Figure 1. Then the area of the triangle is given by
\[
\frac{1}{2}(ad + cf + eb) - \frac{1}{2}(bc + de + fa).
\]
Verify this.

5. Consider the quadrilateral with vertices \((a_1, b_1)\), \((a_2, b_2)\), \((a_3, b_3)\), and \((a_4, b_4)\) below. We can divide it into two triangles by drawing a diagonal from vertex \((a_2, b_2)\) to \((a_4, b_4)\). Use the surveyor’s formula for triangles from Exercise 4 to compute the area of each of the two triangles. Add your two answers together to show that the area of the quadrilateral is given by the surveyor’s formula for quadrilaterals:
\[
\frac{1}{2}(a_1b_2 + a_2b_3 + a_3b_4 + a_4b_1) - \frac{1}{2}(b_1a_2 + b_2a_3 + b_3a_4 + b_4a_1).
\]
Explain what happened to the terms involving the products \(a_4b_2\) and \(b_4a_2\). Your explanation should use the fact that the formula requires the vertices to be oriented in a consistent manner.

---

4The formula is easy to remember: if you list the coordinates in a column (repeating the first pair of coordinates at the bottom of the list), then the positive terms in the formula correspond to “down diagonals” and the negative terms in the formula correspond to “up diagonals”.

5That is, provide a geometric/algebraic argument that this formula gives the area of the triangle. To do this you may use that the area between two curves can be expressed as a definite integral, that the area of a triangle is \(\frac{1}{2}hw\) where \(h\) is a height of the triangle and \(w\) is the corresponding base, that the area of a triangle is given by Heron’s formula as \(\sqrt{s(s-x)(s-y)(s-z)}\) where \(x, y,\) and \(z\) are the side lengths of our triangle and \(s\) is the semiperimeter \(\frac{x+y+z}{2}\), or . . .

6This means that you should provide an algebraic demonstration that the result is valid. Substituting in numbers and checking that they work does not demonstrate that the equality is valid in general.