Most of the following exercises are modified versions of exercises from your text book *Multivariable Calculus* by James Stewart.

59. (14.8) Suppose we are given an ellipse (in black in the figure) and a point \( A \) outside the ellipse. Let \( B \) be a point on the ellipse closest to \( A \). Apollonius (third century BCE) proved that the vector \( \overrightarrow{AB} \) is perpendicular to the tangent line (in blue) to the ellipse at the point \( B \). Use the method of Lagrange multipliers to explain why Apollonius’ result holds. (Hint: the pink circle is a level curve for the function that needs to be minimized.) What can you say about the tangent to the ellipse at a point on the ellipse where the distance to \( A \) is maximized?

60. (14.8) Find the extreme values for the function \( \ln(x^2 + 1) + \ln(y^2 + 1) + \ln(z^2 + 1) \) subject to the constraint \( x^2 + y^2 + z^2 = 50 \).

61. (14.8) Find the extreme values for the function \( 8y - 4x \) subject to the constraints \( y^2 + x^2 - 1 = 0 \) and \( 2x - z - y - 2 = 0 \).

62. (15.1) The integral
\[
\iint_R \sqrt{25 - x^2} \, dA
\]
with \( R = [-3, 1] \times [-1, 4] \) represents the volume of a solid. Sketch the solid.

63. (15.1) Sketch the solid whose volume is given be the iterated integral
\[
\int_3^5 \int_{-1}^4 (27 - 3x - 2y) \, dx \, dy.
\]

64. (15.1) Exercises 6 and 7 in §15.1 of Stewart’s *Multivariable Calculus*.

65. (15.2) In evaluating a double integral over a region \( D \), a sum of iterated integrals was obtained as follows:
\[
\iint_D f(x, y) \, dA = \int_{-2}^0 \int_{-2}^x f(x, y) \, dy \, dx + \int_0^4 \int_{-\sqrt{x}}^x f(x, y) \, dy \, dx.
\]
Sketch the region and express the double integral as an iterated integral with reversed order of integration.

66. (15.2) Do Exercises 46, 48, and 50 of §15.2 in Stewart’s *Multivariable Calculus*.

67. (15.3) Do Exercises 16, 22, and 32 of §15.3 in Stewart’s *Multivariable Calculus*.

68. (Review) The hyperbolic cosine and sine functions are defined by
\[
sinh(x) = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh(x) = \frac{e^x + e^{-x}}{2}.
\]
(a) Compute the derivatives and antiderivatives of $\cosh$ and $\sinh$.

(b) Let $C$ be the space curve with parametric equation $r(t) = (\cosh(t), \sinh(t))$ for $t \in \mathbb{R}$. Show that $C$ is the right branch of the hyperbola with equation $x^2 - y^2 = 1$.

(c) The $\cosh$ function can be used to describe the curve that a rope hanging between two poles assumes (called a catenary, it is also the shape of the St. Louis Arch\textsuperscript{[1]}). Suppose a hanging cable is parameterized by $\ell(x) = (x, 30 \cosh(x/30))$ for $x \in [-38, 38]$ where $x$ is measured in meters. Determine the length of the cable.

\textsuperscript{[1]}As in the movie \textit{Trains, Planes, and Automobiles}. 