1. The vector field shown below

![Vector Field Image](image)

is called a shear layer in fluid mechanics. Use Stokes’ theorem to argue that it has strongly nonzero curl and the divergence theorem to argue that it has zero divergence. Is the direction of the curl into the page or out of it?

*Note:* To apply Stokes, first note that the only component of curl that can be nonzero is the one out of the page. Next consider square loops (square because it is better for the geometry of the flow). Next argue that line integral of the vector field around any such loop is nonzero. Is the line integral positive or negative?

*Note:* To apply the divergence theorem, use the same square loops but now convince yourself that the flux is zero.

**Solt:** The curl is negative everywhere and points into the page.

To deduce that imagine a tiny square placed at a point. Imagine the tiny square with counterclockwise orientation. The line integral of the vector field \( \int F \cdot dr \) (also called the circulation) over that tiny square will be negative. That is because the line integral is zero over the vertical edges \( F \) orthogonal to \( dr \) and the two horizontal edges sum to a negative contribution. By the right hand rule, the curl must be into the page.

The divergence is zero because there is no net fluid entering or leaving the little square along the horizontal edges of the little square. Fluid enters on the left edge at the same rate it leaves on the right edge.

2. Evaluate the flux or surface integral \( \int \int_S F \cdot dS \) where \( F = xy\mathbf{i} + 4x^2\mathbf{j} + yz\mathbf{k} \) over the surface \( S \) given by \( z = xe^y \) with \( 0 \leq x \leq 1 \) and \( 0 \leq y \leq 1 \). The unit normal to the surface points upward.
Soln:

\[ \mathbf{r} = (x, y, xe^y) \]
\[ dS = \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} \, dx \, dy = (-e^y, -xe^y, 1) \, dx \, dy \]
\[ \iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^1 -4x^3e^y \, dx \, dy = 1 - e. \]

3. Suppose \( \mathbf{F} \) is a vector field and \( \int_S \text{curl} \mathbf{F} \cdot d\mathbf{S} = 1 \) where \( S \) is the upper hemisphere \( x^2 + y^2 + z^2 = 1, \, z \geq 0 \), oriented outward. Sketch another surface for which the integral evaluates to the same number.

**Soln:** \( z = 1 - (x^2 + y^2) \) with \( z \geq 0 \). The surface is an inverted paraboloid and it has the boundary as the upper hemisphere. By Stokes's theorem, the flux of the curl through the two surfaces must be the same.

4. Let \( C \) be the curve of intersection of the cylinder \( x^2 + y^2 = 9 \) and the plane \( x + z = 5 \) with positive or counterclockwise sense. Let \( \mathbf{F} = xi + yj + zk \). Evaluate the line integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \) in two ways: by parametrizing the curve of intersection and using Stokes theorem.

**Soln:** The curve is parametrized as \( (x, y, z) = (3 \cos t, 3 \sin t, 5 - 3 \cos t) \), \( 0 \leq t \leq 2\pi \). To evaluate \( \int_C x \, dx + y \, dy + z \, dz \), it is easiest to notice that \( \mathbf{F} = \nabla f \) for \( f = (x^2 + y^2 + z^2)/2 \).

Because \( \mathbf{F} \) is conservative, \( \int_C \mathbf{F} \cdot d\mathbf{r} \) must come out to be 0. Also, \( \text{curl} \mathbf{F} = 0 \).

5. In the vector field sketched below

![Vector Field Diagram](image)

use the divergence theorem to figure out if the divergence is positive or negative at \( P_1 \) and \( P_2 \).

**Soln:** There is flux (or fluid) going into \( P_1 \). The divergence at \( P_1 \) is negative. There is flux (or fluid) going out of \( P_2 \). The divergence at \( P_2 \) is positive.

6. Verify Stokes' theorem for \( \mathbf{F} = -yi + xj - 2k \) and \( S \) being the cone \( z^2 = x^2 + y^2 \), \( 0 \leq z \leq 4 \), oriented downward.
The cone can be described as \( \mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \), with \( z^2 = x^2 + y^2 \), \( z \geq 0 \), and \( 0 \leq x^2 + y^2 \leq 16 \).

Thus, the boundary of the cone is \( x^2 + y^2 = 16 \) with \( z = 4 \). Because the cone is oriented downward, the sense of the boundary is clockwise as you can understand by staring at the picture below and applying the right hand rule:

![Diagram of a cone with its boundary described mathematically]

The boundary is parametrized as \( x = 4 \cos(t) \), \( y = 4 \sin(t) \), \( z = 0 \) with \( t \) decreasing from \( 2\pi \) to 0. So the line integral is \( \int_C -y \, dx + x \, dy = \int_0^{2\pi} 16 (\sin^2 t + \cos^2 t) \, dt = -32\pi \).

Now to calculate the flux, first note that \( \text{curl} \, \mathbf{F} = 2\mathbf{k} \).

Second, calculate the vector area element \( d\mathbf{S} \):

\[
\begin{align*}
\frac{\partial \mathbf{r}}{\partial x} &= \mathbf{i} + \frac{\partial z}{\partial x} \mathbf{k} = \mathbf{i} + \frac{x}{z} \mathbf{k} \\
\frac{\partial \mathbf{r}}{\partial y} &= \mathbf{j} + \frac{\partial z}{\partial y} \mathbf{k} = \mathbf{j} + \frac{y}{z} \mathbf{k}.
\end{align*}
\]

The area element is given by

\[
d\mathbf{S} = \frac{\partial \mathbf{r}}{\partial y} \times \frac{\partial \mathbf{r}}{\partial x} \, dx \, dy = \left( \frac{x}{z} \mathbf{i} + \frac{y}{z} \mathbf{j} - \mathbf{k} \right) \, dx \, dy.
\]

The flux integral becomes:

\[
\int \int_S \text{curl} \, \mathbf{F} \, d\mathbf{S} = \int \int_{x^2+y^2\leq 16} (2\mathbf{k}) \cdot \left( \frac{x}{z} \mathbf{i} + \frac{y}{z} \mathbf{j} - \mathbf{k} \right) \, dx \, dy = \int \int_{x^2+y^2\leq 16} -2 \, dx \, dy = -32\pi.
\]

7. Let \( \mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|} \), where \( \mathbf{r} = xi + yj + zk \). Find \( \text{curl} \, \mathbf{F} \) (also denoted \( \nabla \times \mathbf{F} \)) using Stokes’ theorem. Find \( \text{div} \, \mathbf{F} \) (also denoted \( \nabla \cdot \mathbf{F} \)) using the divergence theorem.

**Soln:** First, \( \text{curl} \, \mathbf{F} = 0 \). Take any circle \( C \) that is centered at the origin. Because \( \mathbf{F} \) is flowing out radially, the circulation \( \int_C \mathbf{F} \cdot d\mathbf{r} = 0 \). Stokes’ theorem tells that circulation is created by curl, or locally in a little loop around every point, and curl is circulation per unit area. Because the circulation is zero for every circle \( C \) centered at the origin, no circulation is being created anywhere by symmetry. (A fully complete argument is a little more involved.) Thus, the curl is zero.

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To evaluate the divergence, consider the flux
\[ \int \int_{S_a} \mathbf{F} \cdot d\mathbf{S} \]
with \( S_a \) being the sphere of radius \( a \) centered at zero. Because the flow is radially outward
\[ \mathbf{F} \cdot d\mathbf{S} = |\mathbf{F}| |d\mathbf{S}| = |\mathbf{F}| dS = dS. \]
Thus, the flux equals the surface area of \( S_a \) which is \( 4\pi a^2 \).
The divergence theorem tells us that divergence is flux per unit volume or that flux is created by divergence.
Consider the surfaces \( S_a \) and \( S_{a+da} \). The flux out of the region enclosed by the two surfaces is
\[ 4\pi(a + da)^2 - 4\pi a^2 = 8\pi a da. \]
The volume between the two surfaces is
\[ \frac{4}{3}\pi ((a + da)^3 - a^3) = 4\pi a^2 da. \]
Because of symmetry and because divergence is flux per unit volume, the divergence at a distance \( a \) from the center must be
\[ \frac{8\pi a da}{4\pi a^2 da} = \frac{2}{a}. \]

8. Find the divergence of \( \mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^3} \) using the divergence theorem.

**Solt:** The divergence theorem tells us that divergence is flux per unit volume in a little cell around the point of interest. Flux is created by divergence.
We begin by looking at the sphere \( S_a \): \( x^2 + y^2 + z^2 = a^2 \). To find the flux out of that sphere, first note that
\[ \mathbf{F} \cdot d\mathbf{S} = |\mathbf{F}| |d\mathbf{S}| \]
because \( \mathbf{F} \) and \( d\mathbf{S} \) are both radially outward on the surface of \( S_a \). Next,
\[ |\mathbf{F}| = \frac{|\mathbf{r}|}{|\mathbf{r}|^3} = \frac{1}{a^2} \]
on the surface of sphere. Therefore
\[ \mathbf{F} \cdot d\mathbf{S} = \frac{dS}{a^2}. \]
Integrating over the surface \( S_a \), we get the flux to be \( \frac{4\pi a^2}{a^2} \) or just \( 4\pi \).
The flux is \( 4\pi \) for every sphere \( S_a \) regardless of \( a \). Therefore, no flux is being created as the spherical surfaces \( S_a \) get bigger. Thus, the divergence is zero everywhere EXCEPT at the origin. At the origin, the divergence is such as to create a finite flux of \( 4\pi \).

9. Suppose a charge \( q \) is placed at the point \( \mathbf{r}_0 = \mathbf{i} + \mathbf{j}/\sqrt{2} + \mathbf{k}/\sqrt{2} \). The electric field it generates is given by \( \mathbf{E} = q \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \), where \( \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \). Find the flux of this vector field out of the sphere \( x^2 + y^2 + z^2 = a^2 \) for \( a = 1 \) and \( a = 4 \), respectively.
Soln: As in the previous problem, $\text{div } E = 0$ everywhere EXCEPT at $r = r_0$. At $r = r_0$ the divergence of $E$ is such as to generate a flux of $4\pi q$ ($q$ because of the factor in front in the definition of $E$).

Notice that $|r_0| = \sqrt{2}$. Thus $r_0$ is outside of the sphere $x^2 + y^2 + z^2 = a^2$ with $a = 1$. The divergence is zero at every point inside the sphere and the flux out of it must be zero.

If $a = 4$, then $r_0$ is inside the sphere. The divergence is zero at every point inside the sphere EXCEPT $r_0$. At $r_0$ the divergence is such as to generate a flux of $4\pi q$. Therefore the flux out of the sphere is $4\pi q$. 