Lab book for Math 215

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(Please consider not printing this lab manual)
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Version changes

- **Version 1.01**: Added version changes to lab manual; changed link urls for Mathematica notebooks.

- **Version 1.02**: Added a note for how to change plot ranges in Show to question 1 in Lab 2.

- **Version 1.03**: Updated introduction and Lab 1.
Introduction

There is a distinction between what may be called a problem and what may be considered an exercise. The latter serves to drill a student in some technique or procedure, and requires little, if any, original thought... An exercise, then, can always be done with reasonable dispatch and with a minimum of creative thinking. In contrast to an exercise, a problem, if it is a good one for its level, should require thought on the part of the student... It is impossible to overstate the importance of problems in mathematics.


0.0 General info

While solutions to many problems that one encounters in vector calculus boil down to the application of techniques that one masters in calculus I and II, they often involve a lengthier setup and more steps, and they can be more difficult to visualize. The lab problems in this book have been designed to address all of these points. A secondary but also very important goal of these labs is to facilitate group work. More specifically, these labs were designed with the following goals in mind.

• Explore in-depth problems with motivation arising from a variety of disciplines. Teamwork and the use of a computer will be essential.

• Build a support network of other students in the class with whom you can study and discuss homework.

• Many of the topics and definitions encountered in vector calculus have concrete and useful real-world interpretations — we hope to hone your understanding of these through illustrative examples.

• Provide an introduction to the use of Mathematica, which is a very powerful and useful computational software program. It is part of the platform underlying Wolfram Alpha.

• Make extensive use of Mathematica to develop geometric intuition about the course content.
0.1 Teamwork

We can’t emphasize enough how useful teamwork can be if it’s taken seriously. Studies consistently show its usefulness for university study. Successful teamwork consists of all groups members looking over problems ahead of time and bringing rough ideas to the meeting. The working session should proceed with group members exchanging ideas about problems, and ensuring that every group member understands the conclusions that are reached. This often involves providing multiple explanations for the same conclusion, as not all group members will think about the problem in precisely the same way. Every group member thus benefits from having their ideas refined by the rest of group, and by sharing their personal insight with everyone else.

Another benefit of this approach to the labs is that you can continue working as a team outside of labs. Got a midterm coming up? That’s a perfect time to spend an hour or two as a team chatting things over in a coffee shop. Here’s a sample agenda that you could use for such a meeting.

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**Sample coffee shop midterm meeting agenda.**

1. Discuss how you’re feeling about the exam. How seriously should we take this meeting? Should we work hard, or use this time as a friendly de-stresser?
2. What are the important topics for the exam?
3. What sorts of problems might you see on the exam?
4. Discuss what you’re confused about.
5. Explain to others any course content that you think you understand super well, but which hasn’t come up yet in the discussion.

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Finally, not all team members will have the same comfort level with computers and coding. You’ll collectively gain the most by rotating who codes, so that the most confident computer programer doesn’t do all the computer work. In this way everyone will have the chance to be introduced to *Mathematica*, and to learn how to troubleshoot computer software issues. We have done our best to provide you with ample hints about using *Mathematica* in the labs. If something does go wrong, and from time to time it will, do not panic! Your team members will be your first resource for dealing with the issue. If you can’t figure it out quickly as a team, consult the appendices in the back of the book, ask the lab instructor, or else use an internet search engine such as *Google* to find a solution. In fact, *Google* is usually the quickest way to resolve coding issues, say by searching on the exact error message that you receive from *Mathematica* (surrounded by quotation marks, so that *Google* knows to treat the entire message as a whole). See Appendix B for more tips on fixing broken code quickly.

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1It’s not a bad idea to keep an agenda even for informal meetings like this. If everyone shows up well-prepared and you’d just like to blow off steam, then you can take the agenda less seriously. But it’s useful to have it handy in case you really would like to get work done. Meetings tend to become more social in the absence of a clear agenda.
0.2 Structure of the labs

Each lab spans two weeks. The first session will be called Part A, while the second session will be called Part B. Part A of the lab will cover material from the lab book that you’re reading now. It contains more material than can be covered in a fifty-minute session. At the end of Part A your instructor will distribute four extra problems and assign a subset of them for you to solve before the next lab. The next lab session (Part B) will consist of each team member explaining their solution to the rest of the team.

The lab component of the course comprises five percent of your final grade. You will receive a grade out of ten points for each of Part A and Part B, for a total of at most twenty points per lab. All team members in a group will receive the same grade.

In Part A you will record a single set of solutions to the lab problems and submit them at the end of the session. It is okay if these solutions are somewhat unpolished, but they should make it clear that your group has engaged with the lab material. Your grade for Part A will be based on participation (e.g. were you on task, did you explain solutions to the rest of the class when asked, etc) and your submitted solutions.

At the end of Part B, each student will be required to submit the notes that they used for the presentation of their solution to their team members. Grades will again be based on participation and submitted notes. Emphasis will be placed on participation, and all group members will receive the same grade. It is important that the presenter not simply stand up, recite their solution, and then sit down without ensuring that everybody is happy with the result. The goal of these labs is to communicate mathematics, not necessarily to accrue points via correct solutions (although correct solutions are desirable!). If the outcome of a stimulating discussion is that the team feels that the solution is not quite adequate, no worries! The positive discussion will be reflected in the grade more strongly than the incorrectness of the argument. The presenter should append a summary of the discussion of their solution to their notes, so that both they and the lab instructor have a brief record of it.

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2The exception to this rule is missing team members, who will receive a grade of zero on that part of the lab without impacting the grade of the other team members.

3The previous footnote applies here as well.
0.3 Overview of the labs

It will not be possible for your lab instructor to cover all of the problems in this lab book. You should use the problems that are not covered as study material for your exams.

Lab 1

The first lab covers essential introductory material on vectors and geometry. The material corresponds to section 12.1 through 12.5 of the course text. Depending on timing, your lecture section may not have covered equations of planes before Lab 1A. In this event, don’t worry about problem 4 until after you’ve covered the material. In this lab you will:

• learn how to define and do simple algebraic manipulations with vectors in Mathematica
• visualize the parallelogram law of vector addition
• practice simple algebraic manipulations with vectors
• practice with simple plotting in Mathematica
• explore the geometry behind the dot and cross products. By the end of the lab it is hoped that if you were shown a graphic of some vectors, you could estimate the dot and cross product without doing an algebraic computation
• determine equations of planes
• explore the relationship between the planes defined by various sets of vectors (e.g. sets which are related by algebraic operations)
• have fun!

Lab 2

The second lab covers material on multivariate functions. The material corresponds roughly to sections 13.1 through 13.4, 14.1 and 14.2 of the course text. Your lecture section may not cover section 14.2 on continuity, but there is a guided exercise on this topic in the lab book. In this lab you will:

• Practice with plotting parameterized objects (e.g. lines and surfaces) in Mathematica
• explore the geometric meaning of the curvature of a single-variable function
• learn about the relationship between position, velocity and acceleration (regarded as vector functions)
• use the graphing capabilities of Mathematica to explore the continuity, or lack thereof, of some functions
• have fun!
Lab 3

The third lab covers material on Lagrange multipliers. The material corresponds roughly to sections 14.6 through 14.8 of the course text. Note that your lecture section might not cover multiple constraints (see the end of section 14.8), but this important topic is covered in the lab in the form of a guided problem. In this lab you will:

- use the graphing capabilities of Mathematica to gain an intuitive understanding for why the method of Lagrange multipliers works
- practice with the Mathematica commands N\text{Solve} and \text{Solve} for solving Lagrange multiplier problems
- practice turning optimization word-problems into algebraic problem that you can solve using Lagrange multipliers
- encounter the method of Lagrange multipliers for solving optimization problems subject to multiple constraints
- have fun!

Lab 4

The fourth lab covers changes of coordinates in multiple integrals. The material corresponds roughly to sections 15.1 through 15.9 of the course text. Your lecture section may only covered the general formula for change of coordinates. This goal will use this important result to derive several classical and extremely useful changes of coordinates. In this lab you will:

- derive the change of coordinates formulae for polar, cylindrical and spherical coordinates
- learn how to view the image of regions in the plane under mappings via the Mathematica command \text{ParametricPlot}
- use these changes of coordinates to evaluate some integrals
- explore a bizarre change of variable that is not seemingly motivated by geometry to compute an interesting double integral
- have fun!

Lab 5

The fifth lab covers material on vector fields and the main theorems on integration (Green, Stokes, Divergence) of the course. The material corresponds roughly to sections 16.1 through 16.9 of the course text. It is likely that your lecture section will not have covered Stokes theorem and the Divergence theorem. For this reason this lab has an extra problem on earlier material which will be useful in Lab 5A. In this lab you will:
• learn to plot vector fields in Mathematica using the commands VectorPlot and VectorPlot3D

• explore the geometric difference between conservative and nonconservative vector fields

• use the graphing capabilities of Mathematica to explore the physical meaning of the curl of a vector field

• Use Green’s theorem and the Divergence theorem to evaluate some integrals

• have fun!
1

Vectors and geometry

1.0 Introduction

The aim of this lab is to gently introduce you to both vectors and Mathematica. Please consult the appendices at the back of this lab manual for useful information on using Mathematica. Note that it is possible to access Mathematica from your home through a virtual private network — see Appendix C for more details on this.

1.1 Example 1: the basics of vectors (Stewart, 12.2)

In this example a Mathematica worksheet will guide you through the basics of vector manipulation and some rudimentary plotting. Download the worksheet by clicking on the following link:

http://www.math.lsa.umich.edu/courses/215/lab_content/lab1/vectors.nb

then open it and follow the instructions.

1.2 Example 2: laser tag (Stewart, 12.2)

The aim of laser tag is to shoot players with a laser rifle. Suppose you are standing at the origin of a three-dimensional coordinate system and that the $xy$-plane is the floor. Suppose that a standing opponent has their target at their waist 3 feet above floor level, and that eye level is 5 feet above the floor. There are three opponents. One is standing so that his target is 30 feet along the $x$-axis, another is lying down so that his target is at the point $x = 20$, $y = 15$, and the third lies in ambush so that his target is at a point 8 feet above the point $x = 12$, $y = 30$. Use Mathematica and also confirm by hand the answers to the following questions.

Note that this problem concerns vectors in three-space rather than in the plane. Such vectors are defined simply by adding more coordinates. For example, the code $v := \{1,2,3\}$ defines the vector from the origin of three-space to the point $(1,2,3)$.

---

1This problem is adapted from Calculus: Single and Multivariable by Hughes-Hallett, Gleason, McCallum, et. al.
1. If you aim with your gun at eye level, find the vector from your gun to each of the three targets.

2. If you shoot from waist height, with your gun one foot to the right of the center of your body as you face along the positive \(x\)-axis, find the vector from your gun to each of the three targets.

After you’ve completed these problems, download and open the notebook at the following link:

[http://www.math.lsa.umich.edu/courses/215/lab_content/lab1/lasertag.nb](http://www.math.lsa.umich.edu/courses/215/lab_content/lab1/lasertag.nb)

Follow the instructions in the notebook to check your answer.

### 1.3 Example 3: dot product and cross product (Stewart 12.3, 12.4)

If \(u\) and \(v\) are two vectors with an angle of \(\theta\) between them, then

\[
\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta.
\]

In the first part of this example we will use Mathematica to explore identity (1.1). Open a fresh Mathematica notebook.

1. Write down two vectors \(u\) and \(v\) that have an angle of \(\pi/4\) radians between them.

2. Verify identity (1.1) for your particular \(u\) and \(v\) by hand and using Mathematica. The dot product can be computed using the command \(\mathbf{u} \cdot \mathbf{v}\). The lengths \(\|\mathbf{u}\|\) and \(\|\mathbf{v}\|\) can be computed using the expressions \(\text{Norm}[\mathbf{u}]\) and \(\text{Norm}[\mathbf{v}]\). Finally, \(\cos \pi/4\) can be computed using \(\cos[\pi/4]\).

3. Let \(u = \langle 1, 0 \rangle\), let \(t\) be a real number, and let \(v = \langle \cos t, \sin t \rangle\). Verify that \(v\) has length equal to 1. Use Mathematica to explore what happens to the dot product \(u \cdot v\) as \(t\) varies. For example, you might use the code

\[
\begin{align*}
\mathbf{u} & := \{1,0\}; \\
\mathbf{t} & := 3\pi/7; \\
\mathbf{v} & := \{\cos[t],\sin[t]\}; \\
\text{Graphics}[\{\text{Arrow}[\{\{0,0\},\mathbf{u}\}],\text{Arrow}[\{\{0,0\},\mathbf{v}\}]\}] 
\end{align*}
\]

to define and graph the vectors. In a new cell you can evaluate \(\text{Print}[\mathbf{u} \cdot \mathbf{v}]\). This might simply return the expression \(\cos[t]\). To get a real number out, you can replace \(\mathbf{u} \cdot \mathbf{v}\) with \(\text{N}[\mathbf{u} \cdot \mathbf{v}]\). Vary \(t\) to examine when \(u \cdot v\) is maximized, when it is minimized, and when \(u \cdot v = 0\). Compare your results with formula (1.1).
4. Download and run the worksheet found at the following link

http://www.math.lsa.umich.edu/courses/215/lab_content/lab1/dotproductanimation.nb

to see an animation of part (2). The number that is printed is the value

\[ u \cdot v = \cos t \].

Compare with your results from part (2).

Now we’ll use Mathematica to explore the cross product. If \( u \) and \( v \) are vectors defined in Mathematica, then the cross product \( u \times v \) is computed using the command \( \text{Cross}[u,v] \).

5. Let \( u = \langle 0, 0, 1 \rangle \) and \( v = \langle 1, 0, 0 \rangle \). Compute both \( u \times v \) and \( v \times u \) by hand and using Mathematica. How are these vectors related?

6. Execute the following code in a Mathematica notebook:

\[
\text{u} := \{1,0,0\}; \\
\text{v} := \{1,1,1\}; \\
\text{Graphics3D[}\{\text{Blue, Arrow[}\{\{0, 0, 0\}, \text{u}\}]}, \text{Yellow, Arrow[}\{\{0, 0, 0\}, \text{v}\}], \text{Red, Arrow[}\{\{0, 0, 0\}, \text{Cross[}\text{u}, \text{v}\}]\}]}]
\]

As you vary \( u \) and \( v \), verify the right-hand rule and observe how the cross product changes. Note that \( u \) is graphed in blue, \( v \) is graphed in yellow, and \( u \times v \) is graphed in red. What happens when \( u \) and \( v \) are perpendicular? What if they are colinear? What happens if you rescale \( u \) or \( v \)? Compare the results of your experimentation with the formula

\[ \|u \times v\| = \|u\|\|v\| |\sin \theta|, \]

where \( \theta \) denotes the plane angle between \( u \) and \( v \).

7. Download and run the worksheet found at the following link:

http://www.math.lsa.umich.edu/courses/215/lab_content/lab1/crossproductanimation.nb

to see an animation of part (6).

---

\(^2\)If \( x \) is some real number known exactly, say it’s a multiple of \( \pi \) or something, then the command \( \text{N}[x] \) returns a decimal approximation to the number \( x \).
1.4 Example 4: equations of planes (Stewart 12.5)

This final example explores equations of planes in three-space. Recall that if \( \mathbf{v} = \langle a, b, c \rangle \) is a vector, then \( ax + by + cz = 0 \) is the equation of the plane perpendicular to \( \mathbf{v} \) that contains the origin. If \((x_0, y_0, z_0)\) is a point in three-space, then the equation of the plane perpendicular to \( \mathbf{v} \) that contains \((x_0, y_0, z_0)\) can be found as follows: consider the equation \( ax + by + cz = d \), where \( x, y, z \) and \( d \) are unknowns, and \( a, b, c \) are the coordinates of the normal vector \( \mathbf{v} \). Then the desired value of \( d \) is given by the equation \( d = ax_0 + by_0 + cz_0 \). For example, when \( x_0 = y_0 = z_0 = 0 \) then \( d = 0 \) and this indeed gives the equation \( ax + by + cz = 0 \). This is not the only way to find the equation of a plane in three-space, but it is often the easiest way.

1. Find the equation of the plane \( P_1 \) containing the origin and the endpoints of the vectors \( u = \langle 1, 1, -1 \rangle \) and \( v = \langle -4, 3, 2 \rangle \) using any method that you know. Verify that the three points defining \( P_1 \) do indeed satisfy your equation.

2. The following code will plot the plane \( P_1 \) and the vectors \( u, v \) and \( u \times v \) using the method described in the introduction of this example:

```mathematica
u := {1, 1, -1};
v := {-4, 3, 2};
w := Cross[u, v];
G1 := ContourPlot3D[w.{x, y, z} == 0,
{x, -5, 5}, {y, -5, 5}, {z, -5, 5},
ContourStyle -> Opacity[0.5], Mesh -> False];
G2 := Graphics3D[{Blue, Arrow[{{0, 0, 0}, u}],
Yellow, Arrow[{{0, 0, 0}, v}], Red, Arrow[{{0, 0, 0}, w}]}];
Show[G1, G2]
```

The definition of \( G1 \) describes the plot of the plane, while the definition of \( G2 \) describes the vectors \( u, v \) and \( u \times v \). The command \( \text{Show}[G1, G2] \) plots both graphics in a single plot. Add a new line defining a graphic \( G3 \) defining the plane you found in (1). For example, you could start with the code for the definition of \( G1 \), but replace the equation \( w.{x, y, z} == 0 \) by the equation \( ax + by + cz = 0 \) you found in (1). Once you’ve defined \( G3 \), execute the command \( \text{Show}[G1,G2,G3] \). If your answer for (1) is correct, you should see a single plane and the three vectors \( u, v \) and \( u \times v \).

3. Can you guess the equation of the plane containing the origin and the endpoints of the vectors \( u \) and \( u + v \)? Verify your answer using any method you like.

4. Find the equation of the plane \( P_2 \) containing the endpoints of the vectors \( u, v \) and \( u + v \). Are \( P_1 \) and \( P_2 \) equal? Use Mathematica to plot both planes in a single graphic.
2

Functions of several variables

2.0 Introduction

This lab introduces multivariate functions. One goal of this lab is to become familiar with the graphing capabilities of Mathematica, so that you can easily see the various functions that you’ll be dealing with in this course. We also spend some time discussing the notion of continuity of multivariate functions, which may or may not have been covered in your lecture.

2.1 Example 1: vector functions and space curves
(Stewart 13.1)

In this first problem we use ParametricPlot3D to plot some space curves.

1. Find parametric equations for the line through the origin and in the direction of the vector \( \langle 1, 1, 1 \rangle \).

2. Replace the question marks in the Mathematica command

   \[
   L1 = \text{ParametricPlot3D}\left[\{?, ?, \}, \{t, -1, 1\}\right]
   \]

   with your parametric equation from (1). Execute the command to define the plot of \( L1 \) between \( t = -1 \) and 1. Here \( t \) should be your parameterizing variable.

3. Execute the code

   \[
   L2 = \text{ParametricPlot3D}\left[\{\cos[s], \cos[s], \cos[s]\}, \{s, -1, 1\}, \text{PlotStyle} \rightarrow \text{Yellow}\right]
   \]

   in the same notebook, and then execute \text{Show}[L1, L2] to plot both lines \( L1 \) and \( L2 \) in the same graphic. What do you observe?
4. Plot three circles $S_1$, $S_2$, $S_3$, centered on the origin and of radius 5, such that the circles are distributed among the 3 coordinate planes. Show all three circles in the same graphic using the Mathematica command `Show[S1,S2,S3]`. [Note, it’s possible that `Show` will only display a small portion of three-space, and some of your plots may not be visible. If this happens, try executing the code]

\[
\text{Show}[S1,S2,S3,\text{PlotRange} \rightarrow \{-5,5\},\{-5,5\},\{-5,5\}]
\]

instead]

5. Torus knots are neat space curves that trace out paths on the surface of a torus (donut). Execute the following code to plot a $(p,q)$-torus knot:

\[
p := 2
\]
\[
q := -3
\]

\[
K = \text{ParametricPlot3D}[[\{2+\cos(qt)\}\cos(pt), (2+\cos(qt))\sin(pt), -\sin(qt)\}, \{t,0,2\pi\}]\]

If you then also execute the code:

\[
T := \text{ParametricPlot3D}[[\{2 + \cos(v)\}\cos(u), (2 + \cos(v))\sin(u), \sin(v)\}, \{u,0,2\pi\}, \{v,0,2\pi\}, \text{Mesh} \rightarrow \text{None}, \text{PlotStyle} \rightarrow \text{Opacity}[.5]]
\]

\[
\text{Show}[K,T]
\]

Mathematica will plot the torus knot on the surface of a translucent torus. Experiment with the values of $p$ and $q$. Can you make a conjecture as to what values of $p$ and $q$ yield a knot that can be bent into a circle without breaking the curve? This is a fun question but don’t spend too much time on it, as it’s not a part of the syllabus. Note that the answer can be found on the Torus knots wikipedia page.

2.2 Example 2: curvature (Stewart 13.3)

In this exercise we graph some functions and their curvature. If the graph of a function of a single variable is defined by the equation $y = f(x)$, then the curvature of the graph is given by the function

\[
\kappa(x) = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}} \quad (2.1)
\]
1. Let \( f(x) = e^x \). Sketch the graph of \( f(x) \) between \( x = -2 \) and \( x = 2 \). Examine formula (2.1) for the curvature \( \kappa(x) \) and make a rough sketch of what you expect the graph of \( \kappa(x) \) to look like.

2. Plot \( f(x) \) and \( \kappa(x) \) in a single graphic using Mathematica. Compare the result with your answer to (1). [Hint: once you’ve defined \( f[x] \) and \( k[x] \), then the line \( \text{Plot}\left\{\{f[x], k[x]\}, \{x, -2, 2\}\}\right\} \) will plot both \( f(x) \) and \( \kappa(x) \) in the same graphic. If your plot is empty then make sure that you wrote \( f[x] = E^x \) rather than \( f[x] = E \). For some reason when you omit the subscript the plot appears empty. Note also that Mathematica uses \( E \) for the constant \( e \).]

3. Repeat steps (1) and (2) with \( f(x) = \cos(x) \).

4. Wikipedia says that, intuitively, the curvature of \( f(x) \) at a point \( x \) measures how much the graph of \( f(x) \) deviates from being a straight line. Do you see this in your plots from (2) and (3)? Explain.

2.3 Example 3: velocity and acceleration (Stewart 13.4)

In this example we reconsider the physics problem from part B of the first lab. As always, do not panic! We’re going to walk you through it step-by-step.

Consider a magnetic field of constant value \( M = \langle 0, 0, -1 \rangle \) in three-space. At time \( t = 0 \) a particle of charge equal to 1 and of unit mass is at the point \( (1, 0, 0) \) and it is experiencing an instantaneous velocity of \( v = \langle 0, 1, 1 \rangle \) at time \( t = 0 \). Let \( p(t) = \langle x(t), y(t), z(t) \rangle \) denote the position vector of the particle at time \( t \), and let \( v(t) = p'(t) \) denote its instantaneous velocity at time \( t \). Thus \( x(t), y(t) \) and \( z(t) \) are functions that give the coordinates of the particle at time \( t \).

1. What are the values \( p(0) \) and \( v(0) \)? [Hint, this is a softball question.]

2. Compute the Lorentz force \( F(t) = v(t) \times M \) experienced by the particle at time \( t \) due to its motion through the magnetic field.

3. By Newton’s laws of physics, force equals mass times acceleration. Thus, the vector function \( F(t) \) is also the acceleration of our particle of unit mass. That is, \( v'(t) = F(t) \). Solve for the function \( p(t) \) using the equation \( v'(t) = F(t) \) and your results from parts (1) and (2). [Hint, if you’re not familiar with differential equations, then to get started you’ll need to know that if \( x(t), y(t) \) are functions satisfying the differential equations

\[
x'' = -y', \quad y'' = x',
\]

then there are real numbers \( a \) and \( b \) such that \( x'(t) = a \cos(t) + b \sin(t) \) and \( y'(t) = -b \cos(t) + a \sin(t) \).]
4. Plot the function \( p(t) \) between \( t = 0 \) and \( t = 10 \) using the `ParametricPlot3D` function in Mathematica.

### 2.4 Example 4: continuity (Stewart 14.2)

In a first calculus course one is often taught that a function \( f \) on the real line is continuous at a point \( a \) if the limit of \( f \) exists at \( a \) and this limit is equal to the value \( f(a) \). More symbolically we’d say that \( f \) is continuous at \( a \) if \( \lim_{x \to a} f(x) \) exists and moreover \( \lim_{x \to a} f(x) = f(a) \). So for example, if \( f \) is continuous at 0 and we look at the values

\[
f(0.1), \quad f(0.01), \quad f(0.001), \quad f(0.0001), \ldots,
\]

they get closer and closer to the value \( f(0) \). We could also “approach the limit from the left” by considering the values

\[
f(-0.1), \quad f(-0.01), \quad f(-0.001), \quad f(-0.0001), \ldots
\]

and we’d still get \( f(0) \). To be super concrete, let’s examine \( f(x) = x^2 \) at \( x = 0 \). Since square real numbers are always positive, both of the sequences above are equal in this case to the sequence

\[
0.01, \quad 0.0001, \quad 0.000001, \quad 0.00000001, \quad 0.0000000001, \ldots
\]

This sequence approaches 0, which is indeed the value of \( f(x) = x^2 \) at \( x = 0 \). Note that to actually detect continuity, we’d need to test the values of \( f \) along every sequence of real numbers that approaches 0, not just our special sequence of 0.1, 0.01, 0.001, etc.

What makes continuity of multivariate functions slightly more complicated is that there are many directions along which one can approach a limit point. For a function to be continuous, the various paths of approach should yield the same value, namely the value of the function at the limit point itself.

One finds the following precise definition of continuity in Section 14.2 of Stewart: let \( f \) be a function of two-variables and let \( (a,b) \in \mathbb{R}^2 \). We say that the limit of \( f(x,y) \) as \( (x,y) \) approaches \( (a,b) \) is \( L \) and we write

\[
\lim_{(x,y) \to (a,b)} f(x,y) = L
\]

if for every real number \( \varepsilon > 0 \) there is a corresponding \( \delta > 0 \) such that if \( (x,y) \) satisfies

\[
0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta,
\]

then \( |f(x,y) - L| < \varepsilon \).

We like to think of this definition in the following way: suppose that you’d like to show that a function \( f \) is continuous at \((0,0)\). Then you have to ensure that you always win at the following game: to begin, your friend hands you a target value \( \varepsilon > 0 \). You might think of \( \varepsilon \) as some incredibly small basketball hoop. Once you’re handed
the target, you’ve got to find a tiny little region near \((0, 0)\) (this is where the \(\delta\) comes in) such that whenever you plug a value from this region into the function \(f\), then you land within a distance of \(\varepsilon\) from the target value \(f(0, 0)\). That is, you’ve got to find an entire little region near \((0, 0)\) of radius \(\delta\) that \(f\) throws inside the \(\varepsilon\)-radius basketball hoop around \(f(0, 0)\). Check out figures 1 and 2 in section 14.2 of Stewart for a nice illustration of the “continuity game”.

In each case below we define the value of \(f(x, y)\) at \((x, y) = (0, 0)\) to be 0.

1. Plot the function \(f(x, y) = \frac{x^2-y^2}{x^2+y^2}\) near \((x, y) = (0, 0)\) using the \texttt{Plot3D} command in \textit{Mathematica}. Does \(f\) appear to be continuous at \((0, 0)\)? [Hint, if you’re having trouble with \textit{Mathematica}, you can change to \textit{Wolfram Alpha} input mode and use natural language for this problem. For example, you might simply type “\texttt{graph (x^2 - y^2)/(x^2 + y^2) near (0,0)}” in a \textit{Mathematica} cell formatted for \textit{Wolfram Alpha} input.]

2. What is the limit of \(f(x, y) = \frac{x^2-y^2}{x^2+y^2}\) at \((0, 0)\) computed along the lines \(y = x, x = 0\) and \(y = 0\)? Is \(f(x, y)\) continuous at \((x, y) = (0, 0)\)? Explain your answer.

3. Consider taking the limit of \(f(x, y) = \frac{x^2-y^2}{x^2+y^2}\) along different paths that approach \((0, 0)\). Examine your plot from (1) to make a guess as to what values can be obtained. Can you prove that your guess is correct?

4. Repeat steps (1) through (3) with the functions

\[
f(x, y) = \frac{y}{x}, \quad f(x, y) = \frac{xy}{x^2+y^2}, \quad f(x, y) = \frac{x^2y^2}{x^2+y^2}.
\]
3

Lagrange multipliers

3.0 Introduction

The goal of this lab is to explore how the method of Lagrange multipliers can be used to maximize multivariate functions subject to multiple constraints. During lecture you have seen how to use Lagrange multipliers to maximize a function subject to a single constraint. The general case of several constraints is conceptually no more difficult, although the computations can become unwieldy as the number of variables and equations increase. In such cases the aid of a computer is essential.

This lab begins with an exploration of level curves. The graphing capabilities of the computer will be used to aid you in gaining an intuitive feeling for why Lagrange multipliers “work”. The second example considers a maximization problem with a single constraint that models a real-world situation. The third example explains how to handle multiple constraints. The fourth and final example looks at how you might maximize the amount of luggage that you can fit on an airline using a single suitcase that is subject to two sizing constraints.

3.1 Example 1: the geometry of Lagrange multipliers (Stewart 14.8)

Michael Rogers of Emory University has written a fantastic Mathematica notebook that explores the geometry of Lagrange multipliers. Click on the following link to open the demonstration:

http://www.math.lsa.umich.edu/courses/215/lab_content/lab3/
TheGeometryOfLagrangeMultipliers.cdf

When the notebook opens you should see two square displays. The smaller display at left shows a birds-eye-view of some blue level curves, a constraint curve colored in green and pink, as well as a black targeting reticle. If you click on the leftmost square the targeting reticle will move. The rightmost square displays the same level curves and constraint curve, but from a different 3d perspective. By clicking and dragging on the right image you can change the perspective. The right square displays three vectors in place of the targeting reticle. The blue vector is the gradient of the function encoded by
the level curves. The yellow vector is the projection of the blue vector on the tangent of the constraint curve. The red vector is normal to the constraint curve, which lies in the $x$-$y$ plane. Note also that the value of the function $f(x, y)$ at the location given by the targeting reticle is displayed in the top left of the rightmost square.

**Exploration:** the Geometry of Lagrange Multipliers.

1. Adjust the view of the right display so that you are looking at the image directly from the side, and so that all four of the local maxima and minima on the constraint curve are visible. What do you notice about the relationship between the colors of the constraint curve and the positions of the local maxima/minima?

2. Move the targeting reticle into one of the green regions on the constraint curve. Shift to a view in the right display such that the three vectors are clearly visible. Observe the value of the function, as well as the direction of the blue vector (the gradient of the function). Now move the targeting reticle slightly in the direction of the blue gradient vector, but not so far that it leaves the green part of the constraint curve. How does the value of $f$ change?

3. Continue to shift the targeting reticle slightly in the direction of the gradient several times until you reach where the green region on the constraint curve turns pink. Observe the gradient and normal vector at each step. What do you notice at the transition point?

4. Repeat steps 2 and 3 with the green region replaced by a pink region.

The previous example considered the function $f(x, y) = \frac{1}{2}(x^2 - xy^2 + 3y)$ subject to the constraint $g(x, y) = 1$, where $g(x, y) = \frac{1}{3}((x^2 + y^2)^2 - 12xy)$. We will now explain how to use *Mathematica* to compute the maxima and minima of $f(x, y)$ on the curve $g(x, y) = 1$.

**Computation:**

1. Open a new notebook in mathematica. In the first cell define the functions $f(x, y) = \frac{1}{2}(x^2 - xy^2 + 3y)$ and $g(x, y) = \frac{1}{3}((x^2 + y^2)^2 - 12xy)$. Execute the command:

   ```mathematica
   NSolve[{D[f[x,y],x] == L*D[g[x,y],x] , D[f[x,y],y] == L*D[g[x,y],y] , g[x,y] == 1}, {x,y,L}, Reals]
   ```

   to find the local maxima and minima for $f(x, y)$ on the constraint curve $g(x, y) = 1$ using the method of Lagrange multipliers. Here $L$ is the multiplier, your $\lambda$ from class (we’ve changed the variable to spare you, and us,
from writing $\lambda$ in Mathematica!). Note that we call the Mathematica command `NSolve` rather than `Solve` since the solutions are quite messy! The difference between these two commands is that `Solve` will try (and succeed) in finding exact but messy solutions, while `NSolve` will return approximate solutions that are easier to express in decimal notation.

### 3.2 Example 2: maximizing utility subject to a cost constraint (Stewart 14.8)

The ad agency Sterling-Cooper-Draper-Pryce (SCDP) wishes to spend $10,000 on freelance work. Three freelancers do regular work for the company. Agent X charges $20 per hour, Agent Y charges $25 per hour, and Agent Z charges $30 per hour. Let $x$, $y$ and $z$ denote the number of hours worked by Agents X, Y and Z, respectively. It turns out that, due to various internal personnel conflicts within SCDP, the utility of the freelance work done for SCDP by these agents is approximated by the function $U(x,y,z) = xyz$.

1. Write down the cost constraint equation for SCDP.

2. Help SCDP maximize the utility $U(x,y,z)$ subject to their cost constraint using Lagrange multipliers and the `Solve` function in Mathematica.

3. Confirm your answer in (2) by hand.

### 3.3 Example 3: multiple constraints (Stewart 14.8)

It is common for an optimization problem to be subject to more than one constraint among the variables. Thankfully the method of Lagrange multipliers generalizes nicely to this situation. In the remaining two examples we will discuss how to maximize a function $f(x,y,z)$ subject to two constraints $g(x,y,z) = 0$ and $h(x,y,z) = 0$. The idea behind the general procedure is already visible at this level of technicality.

In the remainder of this example we set

\[
\begin{align*}
    f(x, y, z) &= x^2 + y^2 + z^2, \\
    g(x, y, z) &= \frac{x^2}{3} + \frac{y^2}{4} + \frac{z^2}{16} - 1, \\
    h(x, y, z) &= x - y + z.
\end{align*}
\]

\(^1\)The \(\mathbb{N}\) stands for numerical approximation.
Maximizing $f$ subject to the two constraints $g$ and $h$ has a nice geometric interpretation that we will explain now.

1. Use *Mathematica* to plot the equations $g = 0$ and $h = 0$ in a single graphic. [Hint, define the two graphs using the `ContourPlot3D` command, and display them both in a single graphic using the `Show` command. Look at part 3 of this example if you’re super stuck!]

In the previous plot you should have observed that $g = 0$ defines an ellipsoid, while $h = 0$ defines a plane. These two equations intersect in an ellipse centered on $(0,0,0)$ that we’ll call $E$. An ellipse has two semiaxes consisting of a major axis and a minor axis. Our goal is to use Lagrange multipliers to solve for the endpoints of the semiaxes of $E$. Observe that the function $f(x,y,z)$ gives the square of the length of the vector $\langle x,y,z \rangle$. If $f$ is restricted to take values on an ellipse centered on 0, then $f$ will attain its maximum values at the ends of the major axis, and it will attain its minmum values at the ends of the minor axis. Thus, by finding the maxima and minima of $f(x,y,z)$ subject to the constraints $g = 0$ and $h = 0$, we will be locating the endpoints of the semiaxes of $E$.

Now for the million dollar question: how do we pull off this constrained maximization problem? If one tries to infer from the case of a single constraint, then one natural guess would be to insist that all three vectors $\nabla f$, $\nabla g$ and $\nabla h$ be parallel. Any solutions to this problem will give critical points of $f$ subject to the constraints $g = 0$ and $h = 0$, but this procedure will not find all of the critical points. Insisting that all three vectors $\nabla f$, $\nabla g$ and $\nabla h$ be parallel is too strict. Instead the idea is to find the points where $\nabla f$ can be expressed as a linear combination of $\nabla g$ and $\nabla h$, that is, the points where the vector $\nabla f$ is the plane that contains $\nabla g$ and $\nabla h$. More precisely, in this case we’d introduce two multipliers $\lambda$ and $\mu$, and our goal is to solve the equations

$$g(x,y,z) = 0,$$
$$h(x,y,z) = 0,$$
$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z).$$

There are now five unknowns: $x$, $y$, $z$, $\lambda$ and $\mu$. How many equations are there? As written above, it looks like there are three, but remember that the equation involving the gradients is in fact a vector equation. Thus, the three coordinates of this vector equation yield three equations, for a total of five equations. So with five equations and five unknowns, we feel like we might have a chance of solving the system of equations!

2. In a *Mathematica* notebook define the functions $f$, $g$ and $h$. Then execute the

---

[2] In the special case when the ellipse is a circle of radius $R$, then $f(x,y,z) = R^2$ is constant and every diameter is both a major and minor axis.
code:

sols :=
Solve[{g[x, y, z] == 0, h[x, y, z] == 0, Grad[f[x, y, z], {x, y, z}] == L*Grad[g[x, y, z], {x, y, z}] + M*Grad[h[x, y, z], {x, y, z}]}, {x, y, z, L, M}]

This defines a set sols containing all of the solutions of the constrained optimization problem. The Mathematica command Grad[f[x,y,z],{x,y,z}] solves for the gradient of f(x,y,z) in terms of the coordinates x,y,z. The L and M appearing above are the two Lagrange multipliers. If you execute sols in a cell on its own then Mathematica will print out the four solutions to the maximization (why are there four?). Note how messy they are! You may prefer to use NSolve instead, as we did in example 1. [Hint, If Mathematica is giving you errors here, try executing the commands Clear[f], Clear[g] and Clear[h] at the top of the sheet. Also make sure that you write your function definitions in Mathematica as f[x_,y_,z_] = x^2 + y^2 + z^2, etc. The underscores are important!]

3. Now that we’ve got the solutions, let’s graph everything. Execute the following code in the same notebook that we’ve been working in:

G1 = Graphics3D[{Red, Sphere[{x, y, z}, .2] /. sols}]
G2 = ContourPlot3D[{g[x, y, z] == 0, h[x, y, z] == 0}, {x, -3, 3}, {y, -3, 3}, {z, -3, 3}]
Show[G1, G2]

The code defining G1 plots a sphere of radius .2 at each of the points in sols, that is, at the critical points of our optimization problem. The plot for G2 graphs both g(x,y,z) = 0 and h(x,y,z) = 0. The Show command then plots everything in a single graphic, as usual. Note that the red spheres do appear at the endpoints of the semi-axes of the ellipse E that is the intersection of the ellipsoid g = 0 and the plane h = 0!

3.4 Example 4: luggage on Byzantine Airlines (Stewart 14.8)

Byzantine Airlines requires that all carry-on luggage be perfectly rectangular, that the sum of the length, width and height of a piece of luggage is no more than 13 palaistes\(^3\), and that one side length is half the size of one of the others.

\(^3\)A palaiste is a unit of length used in the Byzantine empire. One palaiste is about three inches.
1. Write down equations encoding the two size constraints for carry-on luggage on Byzantine Airlines.

2. Use the Solve function of Mathematica to find the maximum volume (in cubic palaistes) that a piece of carry-on luggage on Byzantine Airlines can hold.

3. Confirm your answer in (2) by hand.
4

Jacobians

4.0 Introduction

This lab covers sections from chapter 15 of Stewart. Section 15.9 explains how the substitution rule of single variable calculus generalizes to the multivariate setting. Briefly, assume that we are given a \textit{nice}\footnote{In particular, we must assume that \( T \) is onto, and that \( T \) is one-to-one except possibly on the boundary of \( S \). Eventually your research/studies/career may require that you understand what these words mean. If so, please take Math 451 and 452!} map \( T: S \rightarrow R \) where \( S \) and \( R \) are two subsets of some \( \mathbb{R}^n \). In this course we are mainly intested in single, double and triple integrals, and this means we’ll mainly care about the cases when \( n = 1, 2 \) or \( 3 \).

To recall how to change variables in integrals, let \( u_1, u_2, \ldots, u_n \) be coordinates on \( S \), and express \( T \) in the following form:

\[
T(u_1, u_2, \ldots, u_n) = (x_1(u_1, u_2, \ldots, u_n), x_2(u_1, u_2, \ldots, u_n), \ldots, x_n(u_1, u_2, \ldots, u_n)).
\]

So, each \( x_i \) is a function from the set \( S \) to the real line. The \textit{Jacobian} of \( T \) is the determinant\footnote{In this course you’ll only encounter determinants of 1x1, 2x2 and 3x3 matrices. The definition generalizes to all square matrices in a nice way.} of the matrix whose \((i, j)\)-entry is the partial derivative \( \frac{\partial x_i}{\partial u_j} \). It is denoted by the notation \( \left| \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(u_1, u_2, \ldots, u_n)} \right| \) or \( |\text{Jac}(T)| \). If \( f \) is a continuous function on the set \( S \), then the change of variables formula is given by

\[
\int\cdots\int f(x_1, x_2, \ldots, x_n) dx_1 dx_2 \cdots dx_n = \\
\int\cdots\int f(T(u_1, u_2, \ldots, u_n)) \left| \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(u_1, u_2, \ldots, u_n)} \right| du_1 du_2 \cdots du_n.
\]
\[(x(u, v), y(u, v), z(u, v, w))\] and \(T(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))\):

\[
\int f(x)dx = \int f(x(u)) \frac{dx}{du} du,
\]

\[
\iint f(x, y)dxdy = \iint f(x(u, v), y(u, v)) \det \left( \begin{array}{ccc}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}
\end{array} \right) dudv
\]

\[
\iiint f(x, y, z)dxdydz = \iiint f(x(u, v, w), y(u, v, w), z(u, v, w)) \det \left( \begin{array}{ccc}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}
\end{array} \right) dudvdw
\]

Note that in the examples we’ll follow traditions and use coordinates that are labelled differently from \(u, v\) and \(w\), but don’t let this confuse you!

### 4.1 Example 1: polar coordinates (Stewart 15.4)

This example concerns the change of variable given by the map

\[T(r, \theta) = (r \cos \theta, r \sin \theta).\]

Thus, \(x(r, \theta) = r \cos \theta\) and \(y(r, \theta) = r \sin \theta\). In this example you might wish to introduce the extra variables \(r = u\) and \(\theta = v\) when applying the formulae from the introduction.

1. Give an example of two points \((r, \theta)\) and \((t, \phi)\) that map to the same point under \(T\). Can you find an example where \(r\) and \(t\) are nonzero?

2. Let \(S\) denote the set \(S = \{(r, \theta) \in \mathbb{R}^2 \mid r > 0, \theta \in [0, 2\pi)\}\) and let \(R\) denote the set \(R = \mathbb{R}^2 \setminus \{(0, 0)\}\). Does \(T\) define a one-to-one map from \(S\) onto \(R\)? If \((r, \theta), (t, \phi) \in S\) are such that \((r \cos(\theta), r \sin(\theta)) = (t \cos(\phi), t \sin(\phi))\), then can we conclude that \((r, \theta) = (t, \phi)\)?

3. What is the image of the set \(X = \{(r, \theta) \in \mathbb{R}^2 \mid r \in [0, 1], \theta \in [-\pi/2, \pi/2]\}\) under \(T\)? Check your answer with Mathematica using the command ParametricPlot.

4. Compute the Jacobian of the transformation \(T\). You can use Mathematica for this, but it’s pretty easy to do it by hand as well.

5. Write down the change of coordinates formula for polar coordinates using your answer to (4).

6. Evaluate the following double integral

\[
\int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} \sqrt{x^2 + y^2} dxdy
\]

by hand using your answer to (5). Check your answer with Mathematica using the Integrate command \[^3\] [Hint, this integral is related to Problem 3.]
4.2 Example 2: cylindrical coordinates (Stewart, 15.7)

This example is very similar to the previous one. Define a map $T$ in three-variables via the formula

$$T(r, \theta, z) = (r \cos \theta, r \sin \theta, z).$$

1. Show that the Jacobian of $T$ is the same as the Jacobian obtained in example 1, either by hand, or using Mathematica.

2. Write down the change of variable formula for triple integrals in cylindrical coordinates.

3. A spherical bead with a radius of 13mm has a cylinder of radius 5mm drilled through its center, so that it can be threaded on a thin string. What volume remains in the bead? [Hint, use cylindrical coordinates.]

4.3 Example 3: spherical coordinates (Stewart, 15.8)

This example considers spherical coordinates determined by the map

$$T(r, \theta, \phi) = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta).$$

1. Compute the Jacobian of $T$ by hand.

2. Check your answer to (1) using Mathematica. Recall that if you’ve defined a function $f[a, b, c]$, say, in Mathematica, then its partial derivative with respect to $b$ is given by $\text{D}[f[a, b, c], b]$. The other partials are defined similarly. Matrices are defined using lists of rows. For example, the command $\{\{1, 2\}, \{3, 4\}\}$ defines the 2 by 2 matrix with first row $(1, 2)$ and second row $(3, 4)$. Finally, the determinant is computed using the command $\text{Det}$.

3. Write down the change of variable formula for spherical coordinates using your answer to (1).

4. The earth’s mantle lies between the innermost core and the outermost crust. The crust is $5$km thick, the mantle is $2900$km thick, and the core is $3475$km thick. Approximately what volume is the earth’s mantle? [Hint, approximate the earth by a sphere and use spherical coordinates.]

---

4Technically, the Jacobian is a function, and the domains of the Jacobians in examples 1 and 2 are different. So they can’t possibly be “the same”. But if you embed the plane in three-space in the usual way, then one can make sense of this. It’s really the whole point of cylindrical coordinates.
4.4 Example 4: a change of coordinates that must have come from a fever dream (Stewart, 15.9)

In this example we consider the change of coordinates given by the transformation

\[ T(u, v) = \left( \frac{\sin u}{\cos v}, \frac{\sin v}{\cos u} \right). \]

Thus \( x(u, v) = \frac{\sin u}{\cos v} \) and \( y(u, v) = \frac{\sin v}{\cos u} \).

1. Let \( R \) be the region in the \((u, v)\)-plane bounded by the \( u \) and \( v \) axes, and by the line defined by \( u + v = \frac{\pi}{2} \). What is the image of \( R \) under \( T \)? [Hint, you can use the Mathematica command `ParametricPlot` to check this. For example, the code

\[
\text{ParametricPlot}[(\text{Sin}[u]/\text{Cos}[v], \text{Sin}[v]/\text{Cos}[u]), \{u, 0, \pi/2\}, \{v, 0, (\pi/2) - u\}]
\]

should work.]

2. Try to prove that \( T \) is one-to-one on the region \( R \) if you’ve got enough time and you know what it means to be one-to-one.

3. Show that

\[
\int_0^1 \int_0^1 \frac{1}{1 - x^2 y^2} \, dy \, dx = \frac{\pi^2}{8}
\]

using the change of variables given by \( T \). [Hint, use Mathematica to compute the Jacobian of the transformation \( T \).]

Remark: This might seem like a stupid change of coordinates for a stupid computation, but in fact it’s a wonderful change of coordinates that allows you to do a wonderful computation! To see why, note that one can compute the integral a second way, using the geometric series formula and an argument that enables you to swap summation and integration, as follows:

\[
\int_0^1 \int_0^1 \frac{1}{1 - x^2 y^2} \, dy \, dx = \int_0^1 \int_0^1 \sum_{n \geq 0} x^{2n} y^{2n} \, dy \, dx
\]

\[
= \sum_{n \geq 0} \int_0^1 \int_0^1 x^{2n} y^{2n} \, dy \, dx
\]

\[
= \sum_{n \geq 0} \frac{1}{(2n + 1)^2}
\]

When we combine these two computations of the integral, we deduce that

\[ 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots = \frac{\pi^2}{8}. \]
This can be used to show that
\[
\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6},
\]
where \(\zeta(s)\) denotes the Riemann zeta function. Cool!
5

The theorems of Green, Stokes and Gauss

5.0 Introduction

This lab covers the major theorems of this course. It begins with an exploration of vector fields using Mathematica, and it ends with a derivation of the inverse square law of electromagnetism using Gauss’s Divergence theorem. It is possible that you will not have covered the material necessary for problems 4 and 5 of this lab in your lecture sections — come back to them after you have!

5.1 Example 1: vector fields (Stewart, 16.1)

In this example we’ll get used to plotting vector fields by exploring what it means physically for a vector field to be conservative. Recall that this means the following: a vector field $F$ on $\mathbb{R}^n$ is a function $F: \mathbb{R}^n \to \mathbb{R}^n$, and for each vector $v \in \mathbb{R}^n$ we picture $F(v)$ as a vector emanating from the endpoint of $v$. We say $F$ is conservative if there is a scalar function $f: \mathbb{R}^n \to \mathbb{R}$ such that $F = \nabla f$. The function $f$ is then called a potential for $F$.

Roughly speaking, a conservative field is such that along every closed curve, the average of the vectors along the curve works out to zero. Let’s explore this via the following thought experiment: you’re a mote of dust in the atmosphere. The wind is blowing you around and we can model this by a vector field, given by the direction and force of the wind at each point (for convenience we ignore gravity). Imagine that there is some closed curve in the atmosphere such that along the entire curve the wind blows in the same direction as the curve. If you could find your way onto this curve, the wind would continually blow you around the curve. In the absence of friction from air resistance that is larger than the force of the wind, you’d continue to gather speed and travel along the closed curve over and over. As long as the wind pattern remained the same you’d just speed up faster and faster for eternity. This would clearly violate the law of conservation of energy!

1For a more precise statement, see Theorem 3 in section 16.3 of Stewart.
1. Consider the vector field \( F(x, y) = (-y, x) \). Plot it in Mathematica using the command \texttt{VectorPlot}. Does \( F \) appear to be conservative? If not, describe in words (with justification!) a curve along which the vector field has a nonzero line integral. Then prove that the field is not conservative, either by computing the integral, or using another method of your choosing. If you believe the field \( F \) is conservative, find a potential function for it.

2. Repeat (1) with each of the following vector fields: \( F(x, y) = (x, -y) \), \( F(x, y) = (y, 0) \), and \( F(x, y) = (\sin(x), \cos(x)) \). In the last case ensure that your plot range is large enough before you make your guess (e.g. look at \( x \) and \( y \) between \(-5 \) and \( 5 \)).

5.2 Example 2: kayaking in a river (Stewart, 16.2 and 16.4)

You and a friend have decided to go kayaking in the Huron river, but you’ve only got one car. If you go downstream it might be a pain to walk back with the kayak. You’ve decided to brave the current instead and travel in a large circle, so that you wind up back where you started. You choose a stretch of river that has perfectly parallel straight sides that are two-hundred meters apart. The plan is to travel along a circle of radius one-hundred meters and which is tangent to the banks of the river on both sides. In this example we’ll evaluate the work performed on your kayak by the current.

1. Do you expect the force field exerted by the river to be conservative?

2. Suppose that the force field exerted by the river is given by the function \( F(x, y) = (1 - (y/100)^2, 0) \) for \( y \) values between \(-100 \) and \( 100 \). The horizontal lines \( y = 100 \) and \( y = -100 \) correspond to the shores of the river. Plot the vector field \( F \) for \( x \) and \( y \) between \(-100 \) and \( 100 \) using the Mathematica command \texttt{VectorPlot}. Decide whether or not \( F \) is conservative and prove your assertion.

3. The work done by the river is simply given by the line integral of the vector field \( F \) over the circular contour given by the path of your kayak. Evaluate this quantity by hand. [Hint, use Green’s theorem and a change to polar coordinates].

4. Compare your answer to (3) with your answers to (1) and (2). Are you surprised by the result?

5. Compute the work that would be done by the river if you paddled perpendicularly from shore out one kilometer to the center of the river, then you
paddled one kilometer downstream, then you paddled back to the same shore you began from (again using a direct route perpendicular to shore), and finally you walked back to your car along the shore (where the river does no work) for one kilometer.

5.3 Example 3: the physical meaning of divergence (Stewart, 16.5)

The divergence of a vector field is a scalar function. It measures roughly how much a vector field is expanding or contracting near a point. A little more precisely, imagine a gas where each particle is moving in the direction of the corresponding vector field. Then the divergence is measuring how much the gas is expanding or contracting near the specified point. We will explore this using Mathematica.

Note that

1. What does the vector field $F(x, y, z) = \langle 0, 0, 1 \rangle$ look like? Plot $F$ near 0 using the command `VectorPlot3D`. We recommend restricting $x$, $y$ and $z$ to lie between $-0.1$ and 0.1. Do you expect $\text{Div}(F)$ to be near positive, negative, or zero at $(0, 0, 0)$? Confirm your answer either by hand, or by using Mathematica.

2. Repeat step (1) with the functions $F(x, y, z) = \langle x, y, z \rangle$, $F(x, y, z) = \langle -x, -y, -z \rangle$, $F(x, y, z) = \langle x, 0, 0 \rangle$, $F(x, y, z) = \langle (1 + x)^2, 0, 0 \rangle$.

3. As of August 31, 2014, the entry on divergence in Wikipedia states that “If the divergence is nonzero at some point then there must be a source or sink at that position”. Which of the examples from parts (1) and (2) demonstrate that this assertion is patently false?

5.4 Example 4: the physical meaning of curl (Stewart, 16.5)

The curl of a vector field at some point measures approximately how much the vector field is circulating in a small little neighbourhood of the point. Imagine placing a little pinwheel in the flow of the vector field at the point. If it starts spinning, due to the fact that the field is stronger on one side of the pinwheel than the other, then there is a nonzero curl at that point. This kind of spinning of small (in fact, infinitesimally small) pinwheels is what curl measures.
1. Consider the vector field \( F(x, y, z) = (2x, 0, 0) \). Plot this field using the Mathematica command \texttt{VectorPlot3D}. Look at the output from above so that the arrows are horizontal. If you stick a pinwheel in the flow between two of the arrows, do you expect it to spin, or just move along with the same orientation?

2. Compute \( \text{Curl} \, F \) and compare your answer with part (1).

3. Now let \( F(x, y, z) = (2x, 2x, 0) \). Plot \( F \) using Mathematica. Arrange the viewpoint so that it’s top down and the arrows are again horizontal (so the bounding box will look like a square rotated ninety degrees). Judging from this vantage point, what will happen to a small pinwheel if you insert it into the flow between two of the arrows? Will it spin or just move along?

4. Compute \( \text{Curl} \, F \) and compare your answer with part (3).

5. Next let \( F(x, y, z) = (yz, xz, xy) \) and plot \( F \) using Mathematica. First guess whether or not you expect the curl of \( F \) to be nonzero, then compute \( \text{Curl} \, F \) by hand. Are you surprised by the result?

### 5.5 Example 5: Gauss’s law (Stewart, 16.9)

Warning: the final problem in this lab book is another one on physics! Don’t worry though, it’s not so bad and we’ll walk you through it.

In electromagnetism one hears about Gauss’s law, which roughly states that the electric flux of a field through a closed surface is equal to the total charge enclosed by the surface (at least, as long as we ignore a constant called the electric permittivity). This fact is a consequence of two facts: (1) Gauss’s divergence theorem from calculus and (2) a more primitive physical law that is also sometimes called Gauss’s law. The more primitive law states that the divergence of the electric field is equal to the electric charge density. Let’s not worry about how electric charge density is defined. The key point is that if you integrate the density over some three-dimensional region, then you should get the total amount of charge \( Q \) contained within the region (again, ignoring electric permittivity).

Let’s introduce some notation to make things clearer\(^2\). Let \( V \) be some volume with surface \( S \) and give \( S \) the appropriate outward pointing orientation. Suppose that electric particles of total charge \( Q \) are contained in \( V \). Let \( \mathbf{E} \) denote the electric field that they generate and let \( \rho_V \) denote the electric charge density in \( V \), so that \( \text{Div} \, \mathbf{E} = \rho_V \) by the more primitive form of Gauss’s law. Thus, on the one hand,

\[
Q = \iiint_V \rho_V \, dV = \iiint_V \text{Div} \, \mathbf{E} \, dV.
\]

\(^2\)The hallmark of good notation is that it clarifies things instead of obfuscating them.
5.5. EXAMPLE 5: GAUSS’S LAW (STEWART, 16.9)

On the other hand, Gauss’s Divergence theorem states that
\[ \iiint_V \text{Div} \mathbf{E} \, dV = \iint_S \mathbf{E} \cdot d\mathbf{S}. \]

We thus see that \[ Q = \iint_S \mathbf{E} \cdot d\mathbf{S}, \]
or that the total charge contained in \( V \) is equal to the total flux of \( \mathbf{E} \) through the surface \( S \) of \( V \).

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1. Suppose that a single charged particle of charge \( Q \) is at the origin. Consider a sphere \( S \) of radius \( R > 0 \) also centered on the origin. Explain why the particle must generate an electric field on the sphere of the form
\[ \mathbf{E}(R) = e(R) \frac{\langle x, y, z \rangle}{R}. \]
where \( e(R) \) is some single-variable function that only depends on the radius \( R \). Note that we’ve rescaled by the factor of \( R \) so that \( \|\mathbf{E}(r)\| = e(R) \). [Hint, think about symmetry]

2. We’d like to solve for the value \( e(R) \) using Gauss’s law. Let \( S' \) be one half (a hemisphere) of \( S \). Explain why
\[ \iint_{S'} \mathbf{E} \cdot d\mathbf{S} = Q \frac{1}{2}. \]
[Hint, use symmetry and Gauss’s law for the integral over both hemispheres.]

3. Parameterize \( S' \) and evaluate the surface integral \( \iint_{S'} \mathbf{E} \cdot d\mathbf{S} \). [Hint, a change to polar coordinates may help. Use Mathematica if your algebra gets too messy (although one can do it without too much mess)]

4. Deduce that \( e(R) \) is proportional to \( \frac{1}{R^2} \). This is the inverse square law for electric fields.

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3... as long as we ignore electric permittivity
Appendix A

Common Mathematica commands

Remember that Mathematica is case-sensitive. The links below will take you to the corresponding Mathematica reference entry, where you can see what input the function expects, and also some examples of its use.

- `.` — Computes the dot product of two vectors, or the product of two matrices.
- `Arrow` — Create an arrow graphic. Click on the link to see the various ways to use this command. Note that once the graphic is created, it must be displayed using another command, such as `Graphics`.
- `Clear` — clears the values and definitions for a variable or function. Very useful when code stops working!
- `ContourPlot3D` — Creates a 3D graphics plot of an equation. This works even if you cannot solve for a dependent variable, e.g., the equation $x^2 - y^4 + 2xyz^2 = 0$.
- `Cos` — Compute the cosine of a value. The input should be in radians.
- `Cross` — Compute the cross product of two vectors.
- `D` — Compute the partial derivative of a function with respect to some variable. It’s quite robust and can even compute second, third, fourth, etc, derivatives without needing to call $D$ multiple times.
- `Dot` — Computes the dot product of two vectors, or the product of two matrices.
- `Det` — Compute the determinant of a square matrix. While you’ve only seen determinants of matrices up to size 3x3, this function works in general.
- `E` — The numerical constant $e = 2.71828\ldots$
- `Grad` — Compute the gradient of a function. By default it uses Cartesian coordinates, but other coordinate systems (e.g. cylindrical) may be used. See the documentation for details.
- `Graphics` — Creates a two-dimensional graphics object. Note that the result of this is a variable of sorts. It can be displayed by using the `Show` command. Note that the graphic will also be displayed upon creation, unless you define it using the `:=` operator instead of the `=` operator.
• **Graphics3D** — Creates a three-dimensional graphics object. Note that the result of this is a variable of sorts. It can be displayed by using the `Show` command. Note that the graphic will also be displayed upon creation, unless you define it using the `:=` operator instead of the `=` operator.

• **Integrate** — A robust command that can compute both definite and indefinite integrals, as well as multiple integrals. Click on the link to see a bunch of examples.

• **N** — Computes a decimal approximation of a value.

• **Norm** — Returns the length of a vector.

• **NSolve** — Attempts to solve a system of (possibly nonlinear) equations. Any answers are returned as decimals. Also note that this will return *complex* solutions if they exist. To ignore those, include the keyword `Reals` in your function call. For example, `NSolve[x^2==-1,x,Reals]` returns nothing, while `NSolve[x^2==-1,x]` returns two complex numbers.

• **ParametricPlot** — Creates a two-dimensional parametric plot of a curve or region.

• **ParametricPlot3D** — Creates a three-dimensional parametric plot of a curve, surface, or region.

• **Plot** — Creates a plot of one or more functions of a single dependent variable.

• **Plot3D** — Creates a plot of one or more functions of two dependent variables.

• **Pi** — The numerical constant $\pi = 3.14159\ldots$.

• **Print** — prints an expression as output. Useful for displaying the value of a variable.

• **Show** — displays graphics objects on screen. E.g. if you’ve defined multiple plots, suppose they’re called `P1` and `P2`, then you can display them both in a single graph using `Show[P1,P2]` (assuming they’re both of the same type).

• **Sin** — Compute the sine of a value. The input should be in radians.

• **Solve** — Attempts to solve a system of (possibly nonlinear) equations. Exact answers are returned if possible, and they can be quite messy. Also note that this will return *complex* solutions if they exist. To ignore those, include the keyword `Reals` in your function call. For example, `Solve[x^2==-1,x,Reals]` returns nothing, while `Solve[x^2==-1,x]` returns two complex numbers.

• **VectorPlot** — Plots a two-dimensional vector field.

• **VectorPlot3D** — Plots a three-dimensional vector field.
Appendix B

Quick and dirty tips for using Mathematica

The following tips refer to version 10 of Mathematica. Most will likely work as is for earlier versions, too. You may wish to print out this page.

1. **Mathematica will offer auto-complete suggestions.** Suppose that you don’t know how to take a cross product. If you type “Cr” in an empty cell, a list of suggestions will pop down, the first of which is “Cross”. This can be a useful way to find interesting commands!

2. Typing a question mark before a command will make Mathematica display information about the following command. E.g. executing the line “? Prime” makes Mathematica print the message “Prime[n] gives the nth prime number”.

3. **Use \ast to be absolutely sure you’re multiplying correctly.** Suppose that you’ve defined variables a and b. Then you might execute the code ab in the hopes of computing the product, but Mathematica treats this as a single chunk referencing a new variable called ab. To multiply, execute a\ast b or a b.

4. **Mathematica is case-sensitive!**

5. **Avoid creating variables with names that are single capital letters.** To avoid this we usually just double the letter, e.g., use the name FF rather than F.

6. **Use underscores when defining functions.** E.g. write f[x_, y_] = x + y rather than f[x, y] = x + y.

7. **The Clear command is your friend.** If you’re having trouble with functions, say they’re f and g, a good first attempt at a fix is to add the lines Clear[f] and Clear[g] before your definitions of f and g.

8. **You can use Wolfram Alpha directly within Mathematica.** To do so, start a new cell that accepts “Wolfram Alpha queries” rather than Wolfram language input.

9. If Mathematica gives an error, Google the exact error message. Often the first result will tell you how to fix things. These website are also super helpful:
• http://reference.wolfram.com
• http://mathematica.stackexchange.com/

10. **Save your notebooks.** You never know when you’ll want to go back to old code. Also, save frequently in case Mathematica crashes.

11. **If a function call returns terrible looking output, try adding an N to the start of the command and re-execute the code.** For example, if Solve returns ugly answers, try NSolve instead.

12. **The Show command allows you to display multiple graphics in a single picture.** Look at the code in the first lab for examples of this.

13. **Palettes can be useful for beginning users.** Look for the Palette menu at the top of the screen, and choose the lists of commands that you’ll use frequently. Mathematica will then pop-up a menu with a bunch of commands that you can use via pointing and clicking.

14. **Use = over := unless you’re certain of what you’re doing.** This will sometimes result in faster code, and will almost never have a downside.
Appendix C

Off-campus access of Mathematica

Mathematica should be available in most computer labs on campus. There are also several ways to access Mathematica when you’re not on campus.

- **SSH (secure shell):** this allows you to access a university machine remotely. If you want to transfer files back and forth you’ll need to use SFTP (secure file transfer protocol). Instructions for both can be found by clicking on [this link].

- **UMich virtual sites:** this allows you to access a university machine remotely through a web-browser. To use this method, follow the steps:
  1. Click on [this link].
  2. Click on the yellow button that says “Connect Now” in the pop-up window.
  3. Enter your uniqname and password in the Weblogin if you haven’t done so already.
  4. In a new window you should see a box titled “Specialty Sites”. Click on the button next to Mplus, and then hit the “Request Connection” button at the bottom of the screen.
  5. A file should download, and when you execute it, you should see a Windows desktop open on your screen. Find Mathematica in the start menu and get to work!

One downside to this method is that there are not many remote access points available for this software, and so at busy times it may not be available.

- **Remote access of CAEN windows environment:** engineering students can obtain remote access to Mathematica via the College of Engineering. For more information, [click here].

- **When all else fails, try Wolfram Alpha.**