LAB 2: FUNCTIONS OF SEVERAL VARIABLES, PART A

1. Objectives for Lab 2

In this lab, we have the following objectives:

• Understand different mathematical descriptions of 2-dimensional surfaces in 3-dimensional space ($\mathbb{R}^3$) and be able to plot such surfaces using MATLAB.
• Understand how continuity works for functions of several variables.
• Develop an intuitive understanding of curvature and be able to perform related computations using MATLAB.

2. Matlab Commands

Some of the MATLAB commands we will use in this lab are the following.

2.1. `ezsurf(f, [xmin, xmax, ymin, ymax])`. This command plots the surface $z = f(x, y)$ for a function $f$ defined symbolically in MATLAB. For example, to plot the surface $z = f(x, y) = x^2 + y^2$ for $-3 \leq x, y \leq 3$, we first tell MATLAB that we will have a symbolic expression $f(x,y)$:

```
>> syms f(x,y)
```

then define $f(x,y)$ with

```
>> f(x,y)=x^2+y^2;
```

and we use `ezsurf` to plot the surface

```
>> ezsurf(f,[-3 3 -3 3])
```

The default is a surface generated using 60 mesh points to get a smoother graph, or to reflect variations in the surface better, you may specify the number of mesh points as a final argument; e.g.,

```
>> ezsurf(f,[-5, 5, -2, 1], 100)
```

2.2. `surf(X, Y, Z)`. This command is also for plotting surfaces, but plots numerically. The vectors $X$ and $Y$ can be vectors giving the values along the $x$- and $y$-axes from which to generate a grid of values, or can be matrices giving the $x$ and $y$ coordinates of each point on the surface. Generally we'll use the latter. The argument $Z$ is the $z$-coordinate on the surface at each point in the $x$ and $y$ domain. This is illustrated below to plot the surface $z = y \sin(x) + \cos(y)$ on $-7 \leq x, y \leq 7$. We first generate vectors for $x$ and $y$,

```
>> xvec = -7:0.25:7; yvec = -7:0.25:7;
```

And then create a full list of $x$ and $y$ values in the two dimensional plane $-7 \leq x, y \leq 7$ with

```
>> [X,Y] = meshgrid(xvec, yvec);
```
This command creates matrices \( X \) and \( Y \) corresponding to every point in the plane at which we want to plot the surface. To find the \( z \)-values at each, we take advantage of MATLAB’s understanding of vectors and matrices and let

\[
\text{>> } Z = Y.*\sin(X) + \cos(Y);
\]

Note that in MATLAB, the operators \(*\), \(/\), etc., are by default vector operators. To make them operate element-by-element, we precede the operator with a period: \(.\ast\), \(./\), etc. Here we want to take each element of \( Y \) times the sine of the corresponding element of \( X \), and so use the \(.\ast\) operator. We can then plot the surface:

\[
\text{>> surf(X,Y,Z)};
\]

2.3. \texttt{plot3(x, y, z)}. For vectors \( x \), \( y \), and \( z \), this plots the points \((x(i), y(i), z(i))\) in three dimensions. For example, to plot the parameterized curve \( \vec{r}(t) = (t \sin(t), \cos(t), \sqrt{t}) \) for \( 0 \leq t \leq 10\pi \), we could first define a range of \( t \)-values to use with

\[
\text{>> } t = 0:0.02:10*\text{pi};
\]

And then plot

\[
\text{>> plot3(t.*sin(t), cos(t), t.^(1/2))};
\]

(Note that we again use the element-by-element operators here.)

2.4. \texttt{axis([xmin, xmax, ymin, ymax])}. Resets the \( x \) and \( y \) domain of the current graph to \( x_{\text{min}} \leq x \leq x_{\text{max}}, y_{\text{min}} \leq y \leq y_{\text{max}} \).

2.5. \texttt{hold on}. Makes all subsequent plot commands appear on the current axes, instead of resetting the graph. Type \texttt{hold off} to stop this behavior.

3. Assignments

Work on the following assignments in lab. You will need the techniques developed here to work on the Part B problems, which are due in the subsequent lab period.


**Exercise 1**: Use \texttt{surf} to graph the quadratic surface defined by

\[
z = x^2 + 3xy - 2y^2.
\]

Try changing the scale and rotating the surface. What quadratic surface is this?

Is the surface you graphed in Exercise 1 continuous? This turns out to be a surprisingly interesting question when we are in three dimensions. Recall for a function \( f(x) \) of one variable, we said that \( f \) is continuous at \( x = c \) if the limits from the left and right equal the same value:

\[
\lim_{x \to c^{-}} f(x) = L = \lim_{x \to c^{+}} f(x).
\]
so that we can say \( \lim_{x \to c} f(x) = L \), and the value of the function matches this limit: \( f(c) = L \). For functions of more than one variable, e.g., the function in Exercise 1, to be continuous at a point \((a, b)\), we must consider the limiting behavior of \( f \) for all points \((x, y)\) near \((a, b)\). That is, we must see what happens to \( f(x, y) \) as the point \((x, y)\) is constrained to be in a smaller and smaller neighborhood of \((a, b)\). We investigate this idea in the following exercises.

Exercise 2: Use `ezsurf` to plot the surface in Exercise 1, hold the graph, and add a point at \((1, 2, f(1, 2))\) using `plot3`. Then use the `axis` command to zoom in on \((x, y) = (1, 2)\). Does the limit \( \lim_{(x, y) \to (1, 2)} f(x, y) \) appear to be well defined? Is \( f(x, y) \) continuous at \((1, 2)\)?

Now let’s consider a problem where continuity isn’t quite so clear! Let

\[
f(x, y) = \begin{cases} 
\sin(xy) & \text{if } (x, y) \neq (0, 0), \\
\frac{1}{2} & \text{if } (x, y) = (0, 0).
\end{cases}
\]

Note that away from \((0, 0)\) we might correctly expect that \( f \) is continuous. However, at \((0, 0)\) it is not clear what will happen.

Exercise 3: Plot the surface \( z = f(x, y) \) in domain small enough to decide whether the function \( f \) is continuous at \((0, 0)\). Observe the behavior of the function as \((x, y)\) approaches \((0, 0)\) from different quadrants.

Exercise 4: By evaluating \( f \) numerically at points that approach \((0, 0)\) along different curves in the \(xy\)-plane, investigate if the limit

\[
(1) \quad \lim_{(x, y) \to (0, 0)} f(x, y)
\]

exists if we consider points along each of the following:

a. along the positive \(x\)-axis. For example, evaluate \( f \) at the points \((0.01, 0)\), \((0.001, 0)\), \((10^{-4}, 0)\), \((10^{-6}, 0)\), etc.

b. along the positive \(y\)-axis.

c. along the line \(y = x\).

d. along the parabola \(y = x^2\).

Do you think the limit in (1) exists? Explain. Is the function \( f \) continuous at \((0, 0)\)? Why? If not, can you redefine the value \( f(0, 0) \) to make \( f \) continuous at \((0, 0)\)?

3.2. Curvature and Parametric Curves. Suppose a rocket is launched so that its position is initially given by

\[
\vec{p}(t) = <0.1t^3, t^3, 2000t^2>.
\]

How much does its path bend as it takes off? One way to answer this question is through the idea of curvature: how fast the curve bends away from its tangent
line. To make this a measure that we can compare between curves, we need to pick a standard parameterization—otherwise we could get a different value of the curvature for the same curve represented in different coordinate systems! The standard we use is parameterization by arc length: we use as our parameter the distance \( s \) that we’ve moved along the curve. Then, if the curve is given by \( \vec{r}(s) \), we define its curvature to be 

\[
\kappa(s) = \frac{|\vec{T}'(s)|}{|\vec{T}(s)|},
\]

where \( \vec{T}(s) = \frac{\vec{r}'(s)}{|\vec{r}'(s)|} \) is the unit tangent to the curve. It turns out (this is derived in [JS, §13.3]) that we can find \( \kappa \) in terms of \( t \) as

\[
\kappa(t) = \frac{|\vec{p}'(t) \times \vec{p}''(t)|}{|\vec{p}'(t)|^3}.
\]

**Exercise 5:** Plot \( \vec{p}(t) \) for \( 0 \leq t \leq 10 \) using `plot3`. Verify that it looks as you would expect.

**Exercise 6:** Find the curvature at each point along your curve. Plot it (using `plot`) to see where it is a maximum. Does this make sense from your three dimensional plot of \( \vec{p} \)? Note, in particular, what happens to the curvature for larger values of \( t \). Why is this?

**References**