Consider the function
\[
g(x, y) = \begin{cases} 
\frac{x^6 + y^6}{x^6 + y^6 + (y - x^2)^2}, & (x, y) \neq (0, 0), \\
0, & (x, y) = (0, 0).
\end{cases}
\]

Observe that this function is defined for all \((x, y) \in \mathbb{R}^2\).

**Exercise 1:** Show analytically that the limit \(\lim_{(x, y) \to (0, 0)} g(x, y)\) is the same along any line \(y = ax, a \in \mathbb{R}\) as well as the line \(x = 0\). (**Hint:** Use L’Hôpital’s rule.) Verify this result by numerical evaluations using MATLAB.

**Exercise 2:** Now plot the surface \(z = g(x, y)\). What do you see? Is \(g\) continuous at \((0, 0)\)? Explain and give a mathematical argument supporting what you see in the plot and your claim. Add graphs of representative space curves \(<x, ax, g(x, ax)>\) to your surface to illustrate your observation in Exercise 1.

**Exercise 3:** Find a curve along which the limit \(\lim_{(x, y) \to (0, 0)} g(x, y)\) is different from that which you found in Exercise 2. Add this to your graph.

**Exercise 4:** Find a function \(h(x, y)\) defined on all \(\mathbb{R}^2\) with \(h(0, 0) = 0\) such that \(\lim_{(x, y) \to (0, 0)} h(x, y)\) is same along any line \(y = ax\) as well as the line \(x = 0\), and along any parabola \(y = bx^2\), but different along the cubic \(y = x^3\). Graph this surface and representative space curves showing the limits that you found.

**Exercise 5:** Bonus! What is the smallest change you would have to make to the function \(g(x, y)\) to ensure that it is continuous at \((0, 0)\)?

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1“Smallest” is, of course, a poorly defined measure here. But a large change would be doing away with the denominator entirely. Can you do something less significant than that which preserves as much of the character of the original function as possible but makes it continuous?