LAB 4: JACOBIANS, PART B

1. Objectives and Expectations for Lab 4, Part B

- Part B of Lab 4 contains less guidance.
- You are expected to answer problems that are more challenging than the ones in Part A.
- The material covered and the assignments in Part A is instructive for completing Part B.

2. Matlab Commands

Some of the MATLAB commands and concepts we will use in this lab are the following:

2.1. Element-wise operators: .* , ./, etc. To perform an operation as a scalar operation on each element of a vector x, we prepend a period to the operator. Thus if we have

```matlab
x = 1:5;
y = x.^3;
```

makes y the vector [1, 8, 27, 64, 125].

2.2. exp(x). MATLAB’s exp(x) (and other) command(s) returns the expected value for the scalar x; if applied to a vector, e.g., xx = [x1, x2], exp(xx) operates element-wise and returns the vector [exp(x1), exp(x2)].

2.3. integral(fun, xmin, xmax). This command is calculates the integral \( \int_{xmin}^{xmax} \) fun dx, e.g.,

```matlab
integral( @(x) exp(-x.^2 + x - 1), 0, 5 )
```

Note that we need to use element-by-element operators in the integrand.

2.4. integral2(fun, xmin, xmax, ymin, ymax). This is the 2-dimensional generalization of integral. By default, it integrates dy dx, so that ymin and ymax are the boundaries for y that may, in general, be functions of x. Thus, to calculate \( \int_0^1 \int_{\sqrt{y}}^{x^2} \frac{1}{1+x^2+2y^2} \) dy dx, we could use

```matlab
fxn = @(x,y) 1/(1 + x.^2 + 2*y.^2); integral2(fxn, 0, 1, @(x) x.^2, 1 )
```

Note that we again use the element-by-element operators in the functions that we use in the integral. If we want to change the limits of integration, so that we are integrating \( \int_0^1 \int_0^{\sqrt{\gamma}} \frac{1}{1+x^2+2y^2} \) dx dy, we have to reverse the order that MATLAB plugs variables into the integrand, either in the call:

```matlab
integral2(@(y,x) 1/(1+sqrt(y).^2+2*y.^2), 0, 1, 0, @(y) sqrt(y));
```
or in the definition of the function handle for the integrand:
>> fxn = @(y,x) 1/(1 + x.^2 + 2*x.^2);
>> integral2( fxn, 0, 1, 0, @(y) sqrt(y) )
2.5.

2.5. integral3(fun, xmin, xmax, ymin, ymax, zmin, zmax). This command computes integral of a function \( f(x, y, z) \) of 3 variables over a region of the form \( x_{\text{min}} \leq x \leq x_{\text{max}}, y_{\text{min}}(x) \leq y \leq y_{\text{max}}(x), z_{\text{min}}(x, y) \leq z \leq z_{\text{max}}(x, y) \). It functions as integral2, but with the additional arguments.

3. Assignments

Exercise 1: Use the "trick" introduced in Lab A to show that
\[
\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}
\]
using polar coordinates. You may want to read Lab A again and understand why the trick works.

Exercise 2: By using the change of variables
\[
x(u, v) = \frac{\sin(u)}{\cos(v)}, \quad y(u, v) = \frac{\sin(v)}{\cos(u)},
\]
show that
\[
\int_{0}^{1} \int_{0}^{1} \frac{1}{1 - x^2 y^2} dy \, dx = \frac{\pi^2}{8}
\]
extactly (by evaluating the integral analytically). Then compute the integral using MATLAB and verify that your results are consistent.

In Part A we introduced the Jacobian for a change of variables for two variables, noting that if \( x = f(u, v) \) and \( y = g(u, v) \),
\[
\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix}
  f_u & f_v \\
  g_u & g_v
\end{vmatrix} = f_u g_v - f_v g_u,
\]
so that the area element for integration is \( dx \, dy = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du \, dv \). For changes of three variables, we have a similar result: if \( x = f(u, v, w) \), \( y = g(u, v, w) \) and \( z = h(u, v, w) \), we have
\[
\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix}
  f_u & f_v & f_w \\
  g_u & g_v & g_w \\
  h_u & h_v & h_w
\end{vmatrix} = f_u \frac{\partial(y, z)}{\partial(u, v, w)} - f_v \frac{\partial(y, z)}{\partial(u, v, w)} + f_w \frac{\partial(y, z)}{\partial(u, v, w)},
\]
and the volume element for integration is \( dx \, dy \, dz = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du \, dv \, dw \).

Exercise 3: For spherical coordinates, we have \( x = \rho \sin \phi \cos \theta \), \( y = \rho \sin \phi \sin \theta \), and \( z = \rho \cos \phi \). Use the formula for the Jacobian in (2)
to verify that you get the volume element you expect for spherical coordinates.

Exercise 4: The Earth’s mantle lies between the innermost core and the outermost crust. The crust is 5 km thick, the mantle is 2900 km thick, and the core is 3475 km thick. Approximate the volume of the Earth’s mantle by integrating a volume integral. Hint: approximate the earth by a sphere and use spherical coordinates. After you obtain your answer analytically, verify with a numerical computation of the integral using MATLAB.

Exercise 5: A spherical bead with a radius of 13 mm has a cylinder of radius 5 mm drilled through its center, so that it can be threaded on a thin string. What volume remains in the bead? Hint: Use cylindrical coordinates. After you obtain your answer analytically, verify with a numerical computation of the integral using MATLAB.

References