LAB 5: THE THEOREMS OF GREEN, STOKES, AND GAUSS,
PART A

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1. Objectives and Expectations for Lab 5, Part A

In Part A of Lab 5, our two main goals are to make use of MATLAB to plot vector fields in two dimensions, and to use MATLAB to see how the fundamental theorem of calculus generalizes to the theorems of Green, Stokes, and Gauss. Specifically, we seek

- to have a better understanding of Fundamental Theorem for Line Integrals, Green’s theorem, Gauss’ Theorem (Divergence Theorem), and Stokes’ theorem.
- to see how these theorems are all generalizations of Fundamental Theorem of Calculus in some sense.
- to become proficient plotting vector fields and using MATLAB to evaluate integrals to explore and use these theorems.

2. Matlab Commands

Some of the MATLAB commands and concepts we will use in this lab are the following:

2.1. Element-wise operators: .*, ./, etc. To perform an operation as a scalar operation on each element of a vector \( x \), we prepend a period to the operator.

2.2. \texttt{integral(f, xmin,xmax)}. This command calculates the integral \( \int_{x_{\text{min}}}^{x_{\text{max}}} f \, dx \), e.g.,
\[
>> \text{integral( @(x) exp(-x.^2 + x - 1), 0, 5 )}
\]
\textbf{Note} that we need to use element-by-element operators in the integrand.

2.3. \texttt{integral2(f, xmin, xmax, ymin, ymax)}. This is the 2-dimensional generalization of \texttt{integral}. By default, it integrates \( dy \, dx \), so that \( y_{\text{min}} \) and \( y_{\text{max}} \) are the boundaries for \( y \) that may, in general, be functions of \( x \). To change the limits of integration, so that we are integrating \( \int_0^1 \int_0^{\sqrt{9(1+x^2+2y^2)}} dx \, dy \), we have to reverse the order that MATLAB plugs variables into the integrand, e.g., with
\[
>> \text{integral2(@(y,x) 1/(1+x.^2+2*y.^2),0,1,@(y) sqrt(y))};
\]

2.4. \texttt{integral3(f, xmin, xmax, ymin, ymax, zmin, zmax)}. This command computes integral of a function \( f(x, y, z) \) of 3 variables over a region of the form \( x_{\text{min}} \leq x \leq x_{\text{max}}, \, y_{\text{min}}(x) \leq y \leq y_{\text{max}}(x), \, z_{\text{min}}(x, y) \leq z \leq z_{\text{max}}(x, y) \). It functions as \texttt{integral2}, but with the additional arguments.
2.5. \texttt{quiver(xvec,yvec,dxvec,dyvec)}. The \texttt{quiver} command plots vectors with components \((dxvec(i),dyvec(i))\) at the points \((x(i),y(i))\). Thus, to plot the vector field \(\mathbf{F} = \langle y \cos(x), y \sin(x) \rangle\), we generate a grid of points in the \(xy\)-plane with

\[
\begin{array}{l}
\text{>> [x,y] = meshgrid(0:.2:2, 0:.2:2);} \\
\text{and then plot the vector field with} \\
\text{>> quiver(x, y, cos(x).*y, sin(x).*y);}
\end{array}
\]

(Note that we again need the element-by-element operators in the calculation of the vector components.)

3. Assignments

Work on the following assignments in lab. You will need the techniques developed here to work on the Part B problems, which are due in the subsequent lab period.

3.1. \textbf{Vector Fields, Curl, and Divergence}. Recall that we may calculate the curl of a vector field in Cartesian coordinates as \(\text{curl} \mathbf{F} = \nabla \times \mathbf{F}\), and the divergence as \(\text{div} \mathbf{F} = \nabla \cdot \mathbf{F}\).

From a practical perspective, we view the curl of the vector field as the tendency of the field to rotate. For example, if you dip a paddle in a fluid flow and the paddle rotates, this means that the point at which the paddle is placed the fluid flow has a nonzero curl: the underlying vector field pushes particles around in an asymmetric way. This has two implications: first, that things will rotate in a field with nonzero curl, and second, that some paths through the vector field could be easier to travel than others.

Similarly, the divergence of a vector field is the magnitude of the degree to which, at any point, the field acts as a source (pushes out from the point) or a sink (pulls in toward the point). Thus a region of a vector field with a positive divergence will have a net flow out of the region.

\textbf{Exercise 1}: Use \texttt{quiver} to plot the following planar vector fields:

\[
\begin{align*}
\mathbf{F}(x,y) &= \langle y, x \rangle, \\
\mathbf{G}(x,y) &= \langle 4x, 4y \rangle, \\
\mathbf{H}(x,y) &= \langle -2y, 2x \rangle.
\end{align*}
\]

Plot for \(-6 \leq x \leq 6\) and \(-6 \leq y \leq 6\) using \texttt{meshgrid} with step-size 0.5. Note that we can consider each of these as a vector field in \(\mathbb{R}^3\) by appending 0 for their third-components:

\[
\begin{align*}
\mathbf{F}(x,y) &= \langle y, x, 0 \rangle, \\
\mathbf{G}(x,y) &= \langle 4x, 4y, 0 \rangle, \\
\mathbf{H}(x,y) &= \langle -2y, 2x, 0 \rangle.
\end{align*}
\]

Thought about as three dimensional vector fields, which have zero curl? Which have zero divergence? Discuss using the graphs why:

\[
\begin{align*}
\text{div} \mathbf{F} &= 0, & \text{div} \mathbf{G} &= 8, & \text{div} \mathbf{H} &= 0 \\
\text{curl} \mathbf{F} &= 0, & \text{curl} \mathbf{G} &= 0, & \text{curl} \mathbf{H} &= \langle 0, 0, 4 \rangle \neq 0.
\end{align*}
\]
3.2. The Fundamental Theorem of Calculus in One Dimension. In first-
semester calculus, we learn that for a function $f$ of a single variable $x$ that is
continuous on the interval $[a, b]$, 
\[ \int_{a}^{b} f(x) \, dx = F(b) - F(a), \quad \text{where } F'(x) = f(x). \]
That is, the value of the definite integral only depends on the values of an anti-
derivative of that function at the endpoints of the interval. We have seen an
immediate generalization of this to line integrals. The Fundamental Theorem
for Line Integrals states that given a conservative vector field $F: \mathbb{R}^2 \to \mathbb{R}^2$,
with the potential $f: \mathbb{R}^2 \to \mathbb{R}$ (that is, $F = \nabla f$—this is what it means for a
vector field to be conservative), its integral along any curve $\gamma$ with the endpoints $P_0(x_0, y_0)$ and $P_1(x_1, y_1)$, as we traverse from $P_0$ to $P_1$, can be computed as:
\[ \int_{\gamma} F \cdot dr = f(x_1, y_1) - f(x_0, y_0). \]
Since $\nabla f = F$, we may consider the potential $f$ to be an antiderivative of the
vector field $F$ whose integral is being computed along the curve $\gamma$, and the value
of the integral is again computed by evaluating the antiderivative (potential) at
the endpoints of the domain of integration. In three dimensions a line integral
can be computed along any curve in $\mathbb{R}^3$, so that the path of integration need
not be confined in a plane.

Exercise 2: Consider the vector field $F(x, y, z) = \langle e^z y^2, 2e^z xy, e^z x y^2 \rangle$.
Evaluate the line integral
\[ \int_{C} F(x, y, z) \cdot dr, \]
where $C$ is the helix parametrized as
\[ x(t) = 2 \cos(t), \quad y(t) = 2 \sin(t), \quad z(t) = \frac{t}{5}, \quad 0 \leq t \leq 5\pi \]
in two ways: first use MATLAB’s integral command to evaluate the
integral directly, and then show that the field is conservative by finding a
potential and evaluate the potential at the endpoints.

3.3. Generalizations of the Fundamental Theorem Calculus in 2 dimen-
sions. There are two generalizations to the Fundamental Theorem to two di-
dimensions. In two dimensions, the integrals are over 2-dimensional bounded open
regions. Boundaries of these regions are closed curves (loops) that play the role
of the endpoints in Fundamental Theorem of Calculus (e.g. it does not make
sense to think of endpoints of a disk; that role is played by its boundary, which
is a circle).

The first of these generalizations is Green’s Theorem. If $P$ and $Q$ are
scalar functions of $(x, y)$ defined in a domain containing an open region $D$ with
continuous partial derivatives, and \( C \) is a closed curve that is the boundary of the open region \( D \),

\[
\int_{D} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy = \oint_{C} P \, dx + Q \, dy,
\]

where \( C \) has counter clockwise (positive) orientation. Now, consider \( P \) and \( Q \) to be the first two component functions of a vector field that has no component on the \( z \)-axis: \( F = \langle P, Q, 0 \rangle \). Then

\[
\text{curl } F = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) k,
\]

where \( k = \langle 0, 0, 1 \rangle \) is the standard unit vector, so that the integral on the left hand side of (1) is equal to the following surface integral:

\[
\int_{D} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy = \int_{D} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \int_{D} (\text{curl } F) \cdot n \, dS,
\]

where \( n \) is the upward unit normal to the region \( D \): \( n = \langle 0, 0, 1 \rangle \). On the other hand, the integral on the right hand side of (1) can be written as:

\[
\oint_{C} P \, dx + Q \, dy = \oint_{C} F \cdot \langle dx, dy, 0 \rangle = \oint_{C} F \cdot dr
\]

Therefore Green’s theorem (1) becomes

\[
\int_{D} (\text{curl } F) \cdot n \, dS = \oint_{C} F \cdot dr,
\]

which says that the surface integral of the curl of a vector field \( F \) over the region \( D \) (a surface in 3 dimensions) is equal to the line integral of the vector field over its boundary \( C \). This result is known as Stokes’ Theorem, and is our second two dimensional generalization of the fundamental theorem of calculus: thinking of \( \text{curl } F \) for a planar vector field \( F \) as some sort of a vector valued derivative, to compute its integral over \( D \) it is enough to compute the line integral of the vector field \( F \) itself along the boundary \( C \) of \( D \). From this perspective, observe that (1) resembles Fundamental Theorem of Calculus.

From a practical point of view, Green’s Theorem (or Stokes’ Theorem) gives us a choice: would you like to compute the line integral or the double integral in (1)? In applications, you will see that one of them tends to be dramatically easier.

**Exercise 3:** Evaluate the line integral

\[
\oint_{C} (10x^6e^x + 2y) \, dx + (x - 2y^2) \, dy
\]

where \( C \) is the circle centered at \((-2, 3)\) with radius 3. First try to parametrize the curve \( C \) and directly compute this integral. Use MATLAB’s
integral command, and consider whether it would be an integral that you could evaluate by hand. Then evaluate this integral using Green’s Theorem.

Exercise 4: Verify Green’s Theorem for the vector field \( \mathbf{F}(x, y) = (-x^2y, xy^2) \) on the closed disk \( x^2 + y^2 \leq 9 \) by computing both of the integrals in (1).

3.4. Generalizing the Fundamental Theorem Calculus in 3 dimensions. As one might expect, the story does not end in 2 dimensions. This is Gauss’ Theorem (also known as Divergence Theorem), which states the following: If we are given a vector field \( \mathbf{F}(x, y, z) \) whose component functions have continuous partial derivatives in a bounded open region \( V \) in \( \mathbb{R}^3 \) with a smooth boundary \( S \) (\( S \) is a closed surface in \( \mathbb{R}^3 \)), then we have

\[
\iint_{V} (\text{div} \mathbf{F}) \, dV = \iiint_{S} \mathbf{F} \cdot \mathbf{n} \, dS.
\]

Here the left hand side is the volume integral of the divergence of \( \mathbf{F} \) and the right hand side is the surface integral of the vector field itself \( \mathbf{F} \) over the boundary of the region \( V \) – the surface \( S \). Again, if we think of divergence of \( \mathbf{F} \), \( \text{div} \mathbf{F} = \nabla \cdot \mathbf{F} \) as some sort of a derivative of the vector field \( \mathbf{F} \), the identity (3) is precisely the Fundamental Theorem of Calculus, where once again the boundary \( S \) plays the role of endpoints in 1 dimensional case.

This theorem is a conservation law. It states that that the outward flux of a vector field (you may think of the vector field as a fluid flow) through a closed surface \( S \) is equal to the volume integral of the divergence over the region \( V \) enclosed by the surface \( S \). Divergence of a vector field at a point describes the magnitude of the source or sink at that point. This means that the sum of all sources (with sinks regarded as negative sources), which is the left hand side in (3), gives the net flux out of a region, which is the right hand side in (3).

In lab part B we will consider these generalizations further.

References