LAB 5: THE THEOREMS OF GREEN, STOKES, AND GAUSS,
PART B

(c)2017 UM Math Dept
licensed under a Creative Commons
By-NC-SA 4.0 International License.

1. Objectives and Expectations for Lab 4, Part B

- Part B of Lab 4 contains less guidance.
- You are expected to answer problems that are more challenging than the ones in Part A.
- The material covered and the assignments in Part A is instructive for completing Part B.

2. Matlab Commands

Some of the MATLAB commands and concepts we will use in this lab are the following:

2.1. **Element-wise operators: .* , ./, etc.** To perform an operation as a scalar operation on each element of a vector x, we prepend a period to the operator.

2.2. **integral(fun, xmin,xmax).** This command calculates the integral \( \int_{xmin}^{xmax} \) \( fun \) \( dx \), e.g.,

\[
\text{>> integral( @(x) exp(-x.^2 + x - 1), 0, 5 )}
\]

**Note** that we need to use element-by-element operators in the integrand.

2.3. **integral2(fun, xmin, xmax, ymin, ymax).** This is the 2-dimensional generalization of integral. By default, it integrates \( dy \) \( dx \), so that \( ymin \) and \( ymax \) are the boundaries for \( y \) that may, in general, be functions of \( x \). To change the limits of integration, so that we are integrating \( \int_{ymin(x)}^{ymax(x)} \int_{xmin}^{xmax} \frac{1}{1+x^2+2y^2} \) \( dx \) \( dy \), we have to reverse the order that MATLAB plugs variables into the integrand, e.g., with

\[
\text{>> integral2(@(y,x) 1/(1+x.^2+2*y.^2),0,1, 0,@(y) sqrt(y))};
\]

2.4. **integral3(fun, xmin, xmax, ymin, ymax, zmin, zmax).** This command computes integral of a function \( f(x,y,z) \) of 3 variables over a region of the form \( xmin \leq x \leq xmax, ymin(x) \leq y \leq ymax(x), zmin(x,y) \leq z \leq zmax(x,y) \). It functions as integral2, but with the additional arguments.

2.5. **quiver(xvec,yvec,dxvec,dyvec).** The quiver command plots vectors with components \( (dxvec(i),dyvec(i)) \) at the points \( (x(i),y(i)) \). Thus, to plot the vector field \( \mathbf{F} = (y \cos(x), y \sin(x)) \), we may generate a grid of points in the xy-plane with

\[
\text{>> [x,y] = meshgrid(0:2:2, 0:.2:2);}
\]

and then plot the vector field with

\[
\text{>> quiver( x, y, cos(x).*y, sin(x).*y );}
\]
3. ASSIGNMENTS

In each of the following, we explore one of the generalizations of the fundamental theorem of calculus by using MATLAB.

**Exercise 1:** Let \( D \) be the triangular region with vertices \((0, 0), (1, 0),\) and \((0, 1)\). This region can be written as
\[
D = \{(x, y) : 0 \leq x \leq 1, \ 0 \leq y \leq 1 - x\},
\]
and let \( C = \partial D \) be the boundary of \( D \) oriented counter clockwise, consisting of 3 oriented line segments:
\[
C_1: x(t) = t, \ y(t) = 0, \ 0 \leq t \leq 1, \\
C_2: x(t) = 1 - t, \ y(t) = t, \ 0 \leq t \leq 1, \\
C_3: x(t) = 0, \ y(t) = 1 - t, \ 0 \leq t \leq 1.
\]
• Compute the line integral
\[
\iint_C \left[ e^x \log(5 + x) + \cos(y) \right] dx + \left[ 4x^2 \sin(x) - e^{\sin(y)} \right] dy
\]
numerically using the MATLAB command `integral`. You should consider
\[
F(x, y) = \left< e^x \log(5 + x) + \cos(y), 4x^2 \sin(x) - e^{\sin(y)} \right>
\]
and
\[
\oint_C F \cdot dr = \int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr + \int_{C_3} F \cdot dr
\]
with the parametrizations \( r(t) = (x(t), y(t)) \) given above.
• Now verify Green’s Theorem
\[
\oint_C F \cdot dr = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy, \quad F = \langle P, Q \rangle,
\]
by computing the double integral on the right hand side using the definition of the domain \( D \). You should use the MATLAB command `integral2`.

**Exercise 2:** Let \( C \) be the curve of intersection of the cylinder \( x^2 + y^2 = 9 \) and \( x + z = 5 \) with positive (counterclockwise) orientation. Let \( F = xi + yj + zk \). Calculate the line integral \( \int_C F \cdot dr \) first by parameterizing the curve \( C \) and evaluating the line integral, using `integral`, and second by applying Stokes’ theorem. Which would be easier by hand? Which is easier numerically?
Exercise 3: Let \( V \) be the "ice-cream cone" region that is the volume bounded between the upward cone \( z = \sqrt{x^2 + y^2} \) and the half-sphere \( x^2 + y^2 + z^2 = 1 \). This region can be determined by the inequalities:

\[
\begin{align*}
-\frac{1}{\sqrt{2}} & \leq x \leq \frac{1}{\sqrt{2}}, \\
-\sqrt{\frac{1}{2} - x^2} & \leq y \leq \sqrt{\frac{1}{2} - x^2}, \\
\sqrt{x^2 + y^2} & \leq z \leq \sqrt{1 - x^2 - y^2}.
\end{align*}
\]

(1)

Let \( S \) be the surface that is the boundary of \( V \). Use the Divergence Theorem

\[
\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_V (\text{div} \mathbf{F}) \, dV,
\]

i.e.,

\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_V (\text{div} \mathbf{F}) \, dV,
\]

to compute the surface integral

\[
\iint_S \left( e^{\sin(y)} z^2 + \sin(x), xy + yz + zx, \cos(x) \log(1 + xy) - 3xz^2 \right) \cdot \mathbf{dS}
\]

numerically (i.e. you should compute the volume integral instead). The MATLAB command for triple integrals is \texttt{integral3} and the inequalities (1) suggest that you should take \( dV = dz \, dy \, dx \). Compare this with the calculations required to evaluate the surface integral. Which is easier (numerically, or by hand)?

References